# Finsbury Technical Manuals.

# AN ELEMENTARY TREATISE
ON
# THE CALCULUS
FOR ENGINEERING STUDENTS

WITH NUMEROUS EXAMPLES AND PROBLEMS WORKED OUT

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# PREFACE TO THE FOURTH EDITION

THE present edition of this book has been almost entirely re-written and greatly enlarged.

The introductory chapter contains a brief outline of those parts of Algebra and Trigonometry which are essential to a sound knowledge of the Calculus. Chapter II contains a few fundamental facts in Co-ordinate Geometry. The equations to a straight line and to the sections of a cone are also given.

A considerable number of examples and problems have been added at the end of each chapter. It is to be hoped that these will prove of interest to the student.

The chapters on Differential Equations have been considerably amplified.

The book is a brief outline of the course of lectures delivered by the author to his advanced students, and should prove helpful to those who are taking up the study of the subject with a view to being able to apply it to practical problems.

The author has consulted the works of Dr. Forsyth, and Byerly and other leading authors, and he wishes to take this opportunity to thank those who have helped him by suggesting problems, etc.

J. G.

*January*, 1914.

# THE CALCULUS

## CHAPTER I

## INTRODUCTORY

In order to attain to a practical knowledge of the Differential and Integral Calculus, it is recommended that the student should make himself acquainted with the following facts in Algebra, Trigonometry, and Coordinate Geometry.

### The Binomial Theorem.

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{\lfloor 2}x^{n-2}y^2 + \dots nxy^{n-1} + y^n$$

On transposing the first term and dividing by $y$ we have

$$\frac{(x+y)^n - x^n}{y}$$

$$= \quad nx^{n-1} + \frac{n(n-1)}{\lfloor 2}x^{n-2}y + \frac{n(n-1)(n-2)}{\lfloor 3}x^{n-3}y^2 + \text{etc.}$$

If $y$ gets smaller and smaller until it becomes indefinitely small, then we get

$$\frac{(x+y)^n - x^n}{y} \quad = \quad nx^{n-1}$$

Since every term involving $y$ and higher powers of $y$ on the right-hand side may be neglected because $y$ is less than any finite quantity. This result should be carefully borne in mind.

## Partial Fractions.

An expression of the form $\dfrac{3x+4}{(x+1)(x+2)}$ is the algebraic sum of two fractions whose denominators are $x+1$ and $x+2$ It is required to find the fractions.

$$\text{Let} \qquad \frac{3x+4}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

where A and B are numbers to be found and which are independent of $x$.

There are two methods of finding A and B which we shall give.

First method: On clearing of fractions we have

$$3x + 4 = A(x+2) + B(x+1)$$

This is an identity; that is, true for all values of $x$ therefore we can give to $x$ any numerical values we please without altering the values of A and B

Let $x = -1$ and we get

$$-3 + 4 = A \qquad \therefore A = 1$$

Again let $x = -2$ and we get

$$-6 + 4 = -B$$
$$\therefore B = 2$$
$$\therefore \frac{3x+4}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{2}{x+2}$$

Second method: $3x + 4 = A(x+2) + B(x+1)$

$$\therefore 3x + 4 = (A+B)x + 2A + B$$

In this identity the coefficients of like powers of $x$ on both sides are equal

$$\therefore A + B = 3$$

and $\qquad 2A + B = 4$

whence $\qquad A = 1$ and $B = 2$

If $a + bx + cx^2 + dx^3 \ldots = A + Bx + Cx^2 + Dx^3 \ldots$ be true for all values of $x$, then $a = A$, $b = B$, $c = b$ etc.

Let $x = 0$ and we have $a = A$

and therefore

$$bx + cx^2 + dx^3 \ldots = Bx + Cx^2 + Dx^3 \ldots$$

On dividing by $x$ we have

$$b + cx + dx^2 \ldots = B + Cx + Dx^2 \ldots$$

Let $x = 0$ and we get $b = B$
Similarly we get $c = C$ and $d = D$ etc.
Suppose we require the component fractions of

$$\frac{x^2 - 3x + 4}{(x^2 + 1)(x - 1)(x + 2)}$$

This expression is the sum of three fractions whose denominators are $x^2 + 1$, $x - 1$, and $x + 2$ If a denominator be a quadratic expression with no rational factors, the numerator must be a general expression of the first degree. If a denominator be an expression of the first degree, then the numerator is a constant. Therefore

$$\frac{x^2 - 3x + 4}{(x^2 + 1)(x - 1)(x + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{x + 2}$$

On clearing of fractions we have

$$\begin{aligned} x^2 - 3x + 4 \\ &= (Ax+B)(x-1)(x+2)+C(x^2+1)(x+2)+D(x^2+1)(x-1) \\ &= (A + C + D)x^3 + (A + B + 2C - D)x^2 \\ &\qquad + (-2A + B + C + D)x + 2C - 2B - D \end{aligned}$$

In this identity we may equate the coefficients of like powers of $x$ on both sides thus :—

$$\begin{aligned} A + C + D &= 0 \\ A + B + 2C - D &= 1 \\ -2A + B + C + D &= -3 \\ -2B + 2C - D &= 4 \end{aligned}$$

On solving these equations we obtain

$A = \frac{3}{5}$, $B = -\frac{6}{5}$, $C = \frac{1}{3}$ and $D = -\frac{14}{15}$

$$\therefore \frac{x^2 - 3x + 4}{(x^2+1)(x-1)(x+2)} = \frac{3x-6}{5(x^2+1)} + \frac{1}{3(x-1)} - \frac{14}{15(x+2)}$$

The method of resolving an algebraic fraction into its partial fractions is essential in certain examples on the Integral Calculus. If we require the integral of

$$\int \frac{(x^2 - 3x + 4)dx}{(x^2+1)(x-1)(x+2)}$$

we must first resolve it into its partial fractions, thus :—

$$\int \frac{(x^2 - 3x + 4)dx}{(x^2+1)(x-1)(x+2)} = \frac{3}{5}\int \frac{(x-2)dx}{x^2+1} + \frac{1}{3}\int \frac{dx}{x-1} - \frac{14}{15}\int \frac{dx}{x+2}$$

The expressions on the right-hand side are easily integrated.

## Trigonometry.

$$\begin{aligned} \text{Sin}\,(A \pm B) &= \sin A \cos B \pm \cos A \sin B \\ \text{Cos}\,(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\[6pt] \text{Sin}\, A + \sin B &= 2 \sin \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) \\ \text{Sin}\, A - \sin B &= 2 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B) \\[6pt] \text{Cos}\, A + \cos B &= 2 \cos \tfrac{1}{2}(A+B) \cos \tfrac{1}{2}(A-B) \\ \text{Cos}\, A - \cos B &= -2 \sin \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A-B) \end{aligned}$$

$$\text{Sin}\,(A + B)\cos B - \cos(A + B)\sin B = \sin(A + B - B) = \sin A$$

$$\text{Cos}\,(A + B)\cos B + \sin(A + B)\sin B = \cos(A + B - B) = \cos A$$

$$\text{Tan}^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a + b}{1 - ab}$$

$$\text{Tan}^{-1} a - \tan^{-1} b = \tan^{-1}\frac{a - b}{1 + ab}$$

$$\frac{\text{Sin}\,\theta}{\theta} = 1$$

when $\theta$ is indefinitely small.

The perimeter of a regular polygon, of $n$ sides, inscribed in a circle of radius $r$ is given by

$$\text{Perimeter} = 2rn\sin\frac{\pi}{n}$$

$$= 2\pi r\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)$$

and when the number of sides is indefinitely increased the perimeter is ultimately equal to the circumference of the circle

$$\therefore \text{Circumference of circle} = 2\pi r\left(\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\right)$$

where $n$ is indefinitely great

$$\therefore \text{Circumference} = 2\pi r$$

since $\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = 1$ when $n$ is very great.

The area of a regular polygon inscribed in a circle is given by the formula

$$\text{Area} = \frac{nr^2}{2} \sin \frac{2\pi}{n}$$

where $n$ is the number of sides and $r$ is the radius of the circle. The expression for the area may be written in the form

$$\text{Area} = \pi r^2 \left( \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} \right)$$

If the number of sides be increased indefinitely the area of the polygon is ultimately equal to the area of the circle and

$$\frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1$$

$$\therefore \text{Area of circle} = \pi r^2$$

If any point O be taken on a horizontal straight line, and if distances measured along this straight line to the right of O be considered positive, then distance measured from O to the left will be considered negative, and all real numbers whether positive or negative will lie on this straight


FIG. 1.

line. Let us assume that if a real number be multiplied by $\sqrt{-1}$ it turns the line representing the number through a right angle counterclockwise. If OA which represents one unit, be multiplied by $\sqrt{-1}$ then according to our assumption it is turned into the position OB therefore the imaginary number $\sqrt{-1}$ is represented by OB If OB

that is $\sqrt{-1}$ be multiplied by $\sqrt{-1}$ it turns OB into the position OC that is $-1$, since $\sqrt{-1}\sqrt{-1} = -1$. Again if OC that is $-1$, be multiplied by $\sqrt{-1}$, OC is turned into the position OD, that is OD represents $-\sqrt{-1}$, since $-1 \times \sqrt{-1} = -\sqrt{-1}$, and if OD

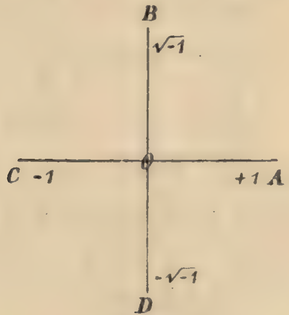


FIG. 2.

or $-\sqrt{-1}$ be multiplied by $\sqrt{-1}$ it turns OD into the position OA that is $+1$, since

$$-\sqrt{-1} \times \sqrt{-1} = +1$$

We see that all real numbers positive or negative lie on a horizontal straight line through O and that any imaginary number such as $4\sqrt{-1}$ lies on a vertical straight line.

A **complex number** is partly real and partly imaginary. $4 + 3\sqrt{-1}$ is a complex number. It can be represented graphically if we measure from O, a length OA equal to 4 units along OX, and at A erect a perpendicular AB equal to 3 units, then OB represents the complex number $4 + 3\sqrt{-1}$ If we had 3 instead of $3\sqrt{-1}$ we should have measured 3 units AB along OX from A but the effect of having $\sqrt{-1}$ multiplied by 3 turns AB into the vertical position. The length OB is

called the **modulus** of the complex number and the angle AOB is called the **amplitude**. OB = 5

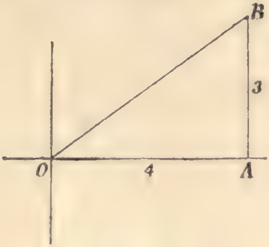


FIG. 3.

Let the angle AOB be denoted by $\theta$

then $\quad OA = OB \cos \theta$

and $\quad AB = OB \sin \theta$

$$\therefore 4 = 5 \cos \theta$$

and $\quad 3\sqrt{-1} = 5\sqrt{-1} \sin \theta$

$$\therefore 4+3\sqrt{-1} = 5(\cos \theta + \sqrt{-1} \sin \theta).$$

$(\cos \theta + \sqrt{-1} \sin \theta)^2$

$$= \cos^2 \theta - \sin^2 \theta + \sqrt{-1} \times 2 \sin \theta \cos \theta$$

$$= \cos 2\theta + \sqrt{-1} \sin 2\theta$$

Similarly

$$(\cos \theta + \sqrt{-1} \sin \theta)^n = \cos n\theta + \sqrt{-1} \sin n\theta$$

and this is true for all values of $n$, positive, negative, integral or fractional.

This theorem may be used to find the $n$ $n$th roots of equations such as

$$x^n = 3$$

$$x^n = 4 + 3\sqrt{-1}$$

Suppose we require the three cube roots of unity. We have to solve the equation

$$x^3 = 1$$

Now $$x^3 = 1 = \cos 2\pi + \sqrt{-1}\sin 2\pi$$
$$= \cos 4\pi + \sqrt{-1}\sin 4\pi$$
$$= \cos 6\pi + \sqrt{-1}\sin 6\pi$$

$$\therefore \quad x = (\cos 2\pi + \sqrt{-1}\sin 2\pi)^{\frac{1}{3}} = \cos\frac{2\pi}{3} + \sqrt{-1}\sin\frac{2}{3}\pi$$

$$\text{and } x = (\cos 4\pi + \sqrt{-1}\sin 4\pi)^{\frac{1}{3}} = \cos\frac{4\pi}{3} + \sqrt{-1}\sin\frac{4\pi}{3}$$

$$\text{and } x = (\cos 6\pi + \sqrt{-1}\sin 6\pi)^{\frac{1}{3}} = \cos 2\pi + \sqrt{-1}\sin 2\pi$$

$$\therefore x = -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}_1$$

$$\text{and } x = -\frac{1}{2} - \frac{\sqrt{3}}{2}\sqrt{-1}_1$$

$$\text{and } x = 1$$

Graphical method of solving the equation

$$x^3 = 1$$

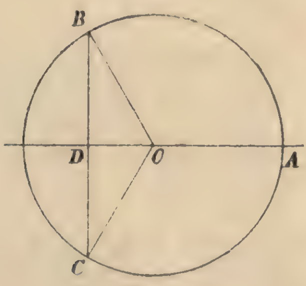


FIG. 4.

Describe a circle ABC of radius equal to unity. Take the points A, B, and C on the circumference 120° apart, Fig. 4, then OA represents the real root of the equation, and OD represents the real parts of the other roots and BD and CD represent the imaginary parts when multiplied by $\sqrt{-1}$, $OD = -\frac{1}{2}$, $BD = \frac{\sqrt{3}}{2}$, and $CD = -\frac{\sqrt{3}}{2}$.

Suppose we require all the roots of the equation

$$x^{12} = 20 = 20 \times 1$$

Here
$$x = \sqrt[12]{20}\,\sqrt[12]{1}$$

Describe a circle with radius equal to $\sqrt[12]{20}$ Fig. 5. Divide the circumference into 12 equal parts AB, BC, CD and

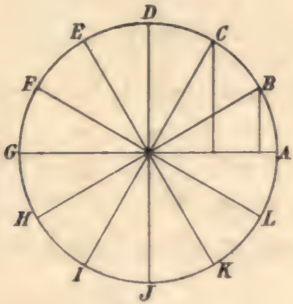


FIG. 5.

so on, then the real parts of the roots are the horizontal projections of the radii and the imaginary parts of the roots are the vertical projections of the radii when multiplied by $\sqrt{-1}$ The angle AOB $=$ 30°, and the root corresponding to the point B is therefore

$$\sqrt[12]{20}\,(\cos 30^\circ + \sqrt{-1}\sin 30^\circ)$$

The root corresponding to the point E is

$$\sqrt[12]{20}(\cos 120^\circ + \sqrt{-1}\sin 120^\circ)$$
$$= \sqrt[12]{20}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}\right)$$

Similarly all the other roots can be found.

Suppose we require the four roots of the equation

$$x^4 = 4 + 3\sqrt{-1}$$

This equation may be expressed in the form

$$
\begin{aligned}
x^4 &= \sqrt{4^2+3^2}\left\{\frac{4}{\sqrt{4^2+3^2}}+\frac{3\sqrt{-1}}{\sqrt{4^2+3^2}}\right\} \\
&= 5\left\{\frac{4}{5}+\frac{3\sqrt{-1}}{5}\right\} \\
&= 5\{\cos\theta+\sqrt{-1}\sin\theta\} \\
\text{where } \cos\theta &= \tfrac{4}{5}, \quad \text{and } \sin\theta = \tfrac{3}{5} \\
\therefore\ x^4 &= 5\{\cos\theta+\sqrt{-1}\sin\theta\} \\
&= 5\{\cos(\theta+2\pi)+\sqrt{-1}\sin(\theta+2\pi)\} \\
&= 5\{\cos(\theta+4\pi)+\sqrt{-1}\sin(\theta+4\pi)\} \\
&= 5\{\cos(\theta+6\pi)+\sqrt{-1}\sin(\theta+6\pi)\} \\
\therefore\ x &= \sqrt[4]{5}\{\cos\tfrac{1}{4}\theta+\sqrt{-1}\sin\tfrac{1}{4}\theta\} \\
\text{and}\qquad x &= \sqrt[4]{5}\left\{\cos\left(\frac{\theta+2\pi}{4}\right)+\sqrt{-1}\sin\left(\frac{\theta+2\pi}{4}\right)\right\} \\
x &= \sqrt[4]{5}\left\{\cos\left(\frac{\theta+4\pi}{4}\right)+\sqrt{-1}\sin\left(\frac{\theta+4\pi}{4}\right)\right\} \\
x &= \sqrt[4]{5}\left\{\cos\left(\frac{\theta+6\pi}{4}\right)+\sqrt{-1}\sin\left(\frac{\theta+6\pi}{4}\right)\right\}
\end{aligned}
$$

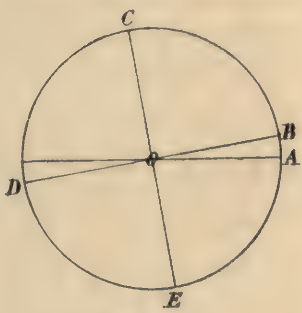


FIG. 6.

Graphically. Describe a circle ABCDE, Fig. 6, of radius equal to $\sqrt[4]{5}$ Set off the angle AOB $= \frac{1}{4}\theta$ and divide

the circumference into four equal parts $\overset{\frown}{BC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DE}$ and $\overset{\frown}{EB}$ Then the horizontal projections of $OB$, $OC$, $OD$ and $OE$ are the real parts of the roots, and the vertical projections of these lines when multiplied by $\sqrt{-1}$ are the imaginary parts of the roots.

# CHAPTER II

## ELEMENTS OF COORDINATE GEOMETRY

The position of a point in a plane is determined if we know its distance from two fixed lines in the plane, the usual convention of signs being understood.

Let OX , OY be two lines in the plane, OX being horizontal and OY vertical.

Horizontal distances to the right and left of O being positive and negative respectively, and vertical distances above and below O being positive and negative respectively. The horizontal distance of a point from the axis of Y is called its *abscissa* and the vertical distance of the point from the axis of X is called the *ordinate* of the point and the two distances are called the coordinates of the point.

**The distance between two points** the coordinates of which are $(x_1y_1)$ and $(x_2y_2)$ (Fig. 7) is given by

$$\textbf{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The area of a triangle the coordinates of its vertices being $(x_1y_1)$, $(x_2y_2)$ and $(x_3y_3)$ is

$$\textbf{Area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

The equation to a straight line parallel to the axis of X is

$$\mathbf{y = c}$$

where $c$ is a constant,

The equation to a straight line parallel to the axis of Y is

$$x = a$$

where $a$ is a constant.

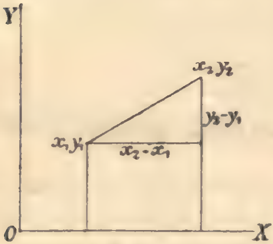


FIG. 7.

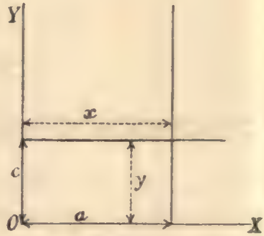


FIG. 8.

The equation to a line passing through the origin inclined to the axis of $x$ at an angle $\theta$ is

$$y = mx$$

where $\tan\theta = m$

If a line does not pass through the origin but cuts the axis of Y $c$ above the origin then its equation is

$$y = mx + c \quad . \quad . \quad . \quad . \quad . \quad (1)$$

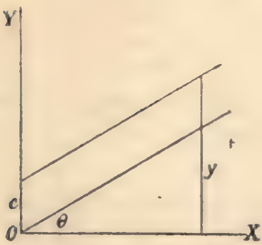


FIG. 9.

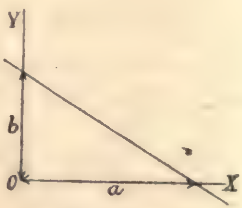


FIG. 10.

This is called the *tangent form* of the equation to a straight line.

The equation to a straight line in terms of its intercepts on the axis of X and Y is

$$\frac{x}{\mathbf{a}} + \frac{y}{\mathbf{b}} = 1 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $a$ is the intercept on the axis of X
and $b$ " " Y

Where $x = 0$, $y = b$, and where $y = 0$, $x = a$

The equation to a straight line may be expressed in the form

$$\mathbf{l}x + \mathbf{m}y = 1 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where $l$ stands for $\frac{1}{a}$, and $m$ is $\frac{1}{b}$

The general form of the equation to a straight line is

$$\mathbf{a}x + \mathbf{b}y + \mathbf{c} = 0 \quad . \quad . \quad . \quad . \quad (4)$$

where $a$, $b$, and $c$ are constants.

Equation to a straight line in terms of the perpendicular on it from the origin and the angle which the perpendicular makes with the axis of X.

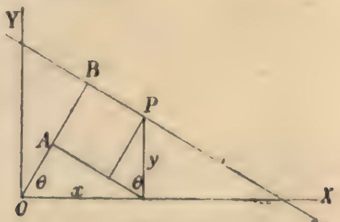


FIG. 11.

Let OB be the perpendicular from the origin on the line, and let this perpendicular make an angle $\theta$ with the axis of X Let the coordinates of P be $(x, y)$ then

$$p = OB = OA + AB$$

$$\therefore p = x \cos\theta + y \sin\theta$$

since $OA = x \cos\theta$ and $AB = y \sin\theta$

The equation is generally written thus

$$\mathbf{x} \cos\theta + \mathbf{y} \sin\theta - \mathbf{p} = 0 \quad . \quad . \quad (5)$$

The equation to a straight line passing through a given point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1).$$

$$y = mx + c \quad . \quad . \quad . \quad . \quad . \quad (a)$$

represents any straight line, and if the point $(x_1, y_1)$ be on this line the coordinates of the point must satisfy the equation

$$\therefore y_1 = mx_1 + c \quad . \quad . \quad . \quad . \quad . \quad (b)$$

and by subtracting $(b)$ from $(a)$ we have

$$y - y_1 = m(x - x_1) \quad . \quad . \quad . \quad . \quad . \quad (c)$$

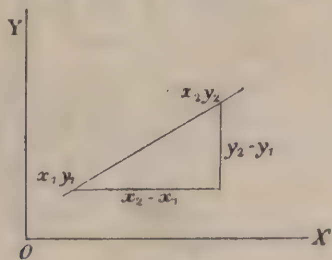


FIG. 12.

If the same line passes through the point $(x_2, y_2)$ we have

$$y_2 = mx_2 + c \quad . \quad . \quad . \quad . \quad . \quad . \quad (d)$$

and by subtracting $(b)$ from $(c)$ we have

$$y_2 - y_1 = m(x_2 - x_1) \quad . \quad . \quad . \quad . \quad (e)$$

on dividing $(c)$ by $(e)$ we have

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

which may be written thus

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad . \quad . \quad . \quad (f)$$

If the points $(x_1 y_1)$ and $(x_2, y_2)$ be indefinitely near to each other then we may write $(f)$ in the form

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

which is the equation to the tangent to any plane curve at the point $(x_1, y_1)$

**The equation to a circle** in its simplest form is $x^2 + y^2 = r^2$ when its centre is at the origin.

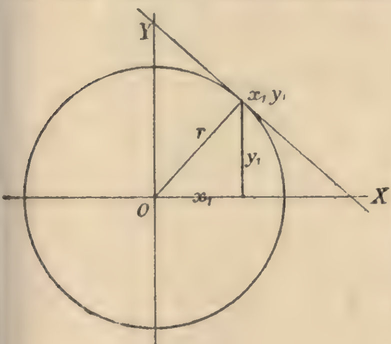


FIG. 13.

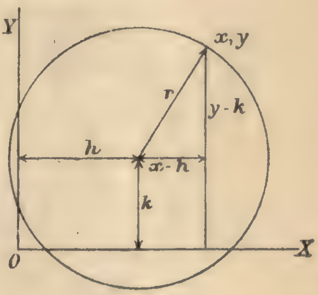


FIG. 14.

Let $(x_1, y_1)$ be any point on the circumference of the circle, then

$$x^2 + y^2 = r^2$$

is the equation to the circle (Fig. 13).

**The equation to the tangent** to this circle at the point $(x_1, y_1)$ is

$$\mathbf{xx_1 + yy_1 = r^2}$$

If the centre of the circle be the point $(h, k)$ then its equation is

$$(x - h)^2 + (y - k)^2 = r^2$$

This equation may be written in the form

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

which is called the *general equation* to a circle. The tangent to this circle at the point $(x_1, y_1)$ is

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

If a cone be cut by a plane in the various ways shown in Fig. 15 we get *two straight lines, a circle, an ellipse, parabola* or *a hyperbola.*

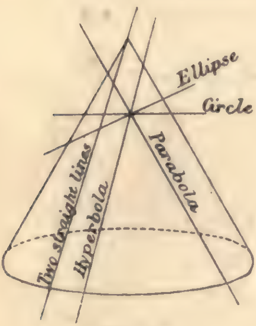


FIG. 15.

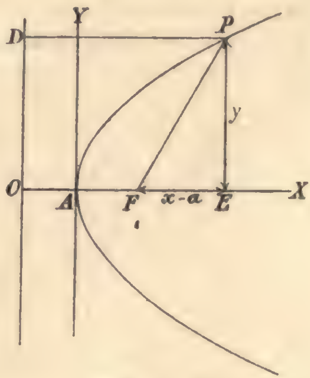


FIG. 16.

**A parabola** is the locus of a point which moves so that its distance from a fixed point called the focus is always equal to its distance from a fixed line called the directrix.

In Fig. 16 let F be the focus and OD the directrix and let AY be the axis of Y, then

$$\begin{aligned} FP^2 &= PD^2 \\ \therefore (x - a)^2 + y^2 &= (x + a)^2 \\ \therefore \mathbf{y^2} &= \mathbf{4ax} \end{aligned}$$

This is the simplest form of the equation to the parabola.

The equation to the tangent to the parabola at the point $(x_1, y_1)$ is

$$yy_1 = 2a(x + x_1)$$

Where the tangent cuts the axis of X, $y = 0$, and therefore $x = -x_1$ This shows (Fig. 17) that

$$OA = OB$$

where OB is taken as the axis of X and where AC is the tangent to the parabola DOC at the point C, O being the origin.

If the cable of a suspension bridge be loaded uniformly horizontally it hangs in a parabolic arc and therefore the

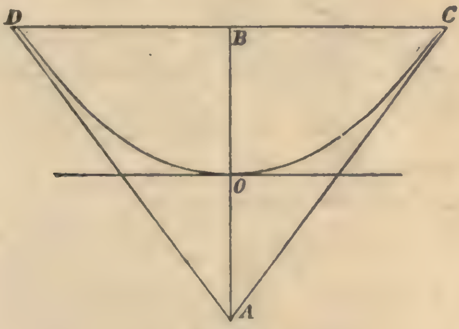


FIG. 17.

tangent at the piers cuts the vertical through O at a point A so that AO = OB

**An ellipse** is the locus of a point which moves so that its distance from a fixed point called the focus bears a constant ratio to its distance from a fixed straight line called the directrix, the ratio being less than unity. The equation to the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Where $a$ and $b$ are the semi-axes

$$b^2 = a^2(1-e^2)$$

$e$ is called the eccentricity.

The equation to the tangent at the point $x_1y_1$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1$$

**A Hyperbola** is the locus of a point which moves so that its distance, from a fixed point, bears a constant ratio to its distance from a fixed straight line, the ratio being greater than unity.

Its equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$$

where the origin is taken midway between the two vertices. The value of $b$ depends upon the constant ratio mentioned in the definition

$$b^2 = a^2(1-e^2)$$

where $e$ stands for the ratio.

The tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$$

at the point $x_1y_1$ is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2} = 1$$

*Examples.*

1. Find the distance between the points $(3, 4)$, $(6 - 2)$

*Ans.* $\sqrt{45}$

2. Find the area of the triangle the vertices being the points $(0, 1)$, $(4, 2)$, and $(1, 6)$ *Ans.* 9·5

3. Find the equation to the line passing through the point $(3, 2)$ making an angle of $120^\circ$ with the axis of $x$.

*Ans.* $y+\sqrt{3}x = 2+3\sqrt{3}$

4. Find the equation to the line the intercepts on the axes of $x$ and $y$ being respectively $4$ and $-3$

*Ans.* $3x - 4y = 12$

5. Find the equation to the line passing through the points $(1, 2)$, $(3, 4)$ *Ans.* $x - y + 1 = 0$

6. Write down the equation to the circle with centre at the origin and radius 6 *Ans.* $x^2 + y^2 = 36$

7. Write down the equation to the circle whose centre is at the point $(2, 3)$ and radius 4

*Ans.* $(x - 2)^2 + (y - 3)^2 = 16$

8. Find the centre and radius of the circle

$$x^2 + y^2 + 8x - 10y + 5 = 0$$

*Ans.* centre $(-4, 5)$, radius $= 6$

9. Find the equation to the tangent to the circle

$$x^2 + y^2 + 4x + 6y - 8 = 0$$

at the point $(2, 3)$ *Ans.* $4x + 6y + 5 = 0$

10. Write down in its simplest form the equation to a parabola, the distance from the vertex to the focus being 2

*Ans.* $y^2 = 8x$

11. Find the equations to the tangents to the parabola $y^2 = 8x$, at the point where $x = 2$

*Ans.* $x \pm y + 2 = 0$

12. Find the equations to the tangents and normals to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ at the point where $x = 2$

*Ans.* Tangents $\frac{2x}{9} \pm \frac{\sqrt{5}}{6}y = 1$

Normals $9\sqrt{5}x \pm 12y - 10\sqrt{5} = 0$

13. Find the equation to the circle passing through the points $(0, 0)$, $(6, 0)$ and $(0, 6)$

*Ans.* $x^2 + y^2 - 6x - 6y = 0$

14. Find the equation to the tangent, at the origin, to the circle $x^2 + y^2 + 6x - 3y = 0$

*Ans.* $6x - 3y = 0$

15. Find the equations to the tangents to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ at the points where $x = 6$

*Ans.* $\frac{6x}{25} \pm \frac{\sqrt{11}y}{20} = 1$

16. The cables of a suspension bridge hang in parabolic arcs when loaded uniformly horizontally. If the span of the bridge be 400 feet and the dip 40 feet, find where a tangent to the curve at the piers cuts the vertical line drawn through the lowest point of the cable.

*Ans.* 40 feet below the vertex.

# CHAPTER III

## RATE

Suppose a motor-car to start from rest and let the distance from the starting point be $s$ miles corresponding to the time $t$ hours, as given in the following table.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $s$ | 0 | 23 | 70 | 143 | 230 | 334 | 400 | 432 | 459 | 515 |
| $t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Let $s$ be plotted vertically and the corresponding time $t$ horizontally; we get a displacement curve on a time base. The average velocity reckoned from the starting point up to any instant is given by:—

$$\text{Av. vel.} = \frac{\text{whole distance}}{\text{time occupied}} \text{ miles per hour}$$

The average velocity during the fifth and sixth hours is the space passed over during that time divided by 2 (the time taken). From the curve or table the space passed over is 170 miles.

$$\therefore \text{ Average velocity } = \tfrac{170}{2} = 85 \text{ miles per hour.}$$

This does not mean that the velocity during the two hours was at the rate of 85 m. p. h.

It may have been much more or less at any particular instant. In order to get the velocity at any particular instant we must draw a tangent to the displacement curve and find the tangent of its inclination to the horizontal.

Suppose we require the velocity at the end of 2·5 hours

from starting. Draw a tangent to the curve (Fig. 18) at the point D, corresponding to 2·5 hours. Then if the

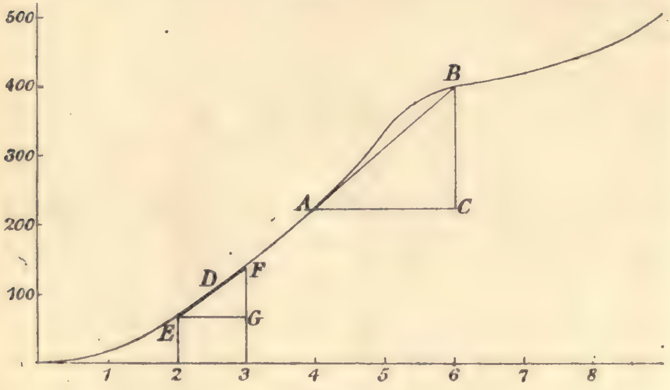


FIG. 18.

velocity had been constant during the third hour, and equal to that at the end of 2·5 hours, the space passed over in one hour would have been 72 miles represented in the figure by FG

$\frac{FG}{EG}$ represents rate of change of displacement at the end of 2·5 hours, but rate of change of displacement is velocity.

Suppose a body to move according to the law $S = t^2$ where $s$ denotes displacement in feet per second and $t$ the time in seconds. The curve is shown in Fig. 19. It is required to find the velocity at any instant. If two points A and B be taken on the curve, then BC $= \delta s$ will represent the space passed over in $\delta t$ of a second where AC represents the small interval of time. The average velocity during the time $\delta t$ is given by

$$\frac{BC}{AC} = \frac{\delta s}{\delta t} = \text{av. vel.}$$

But this does not represent the velocity at the end of $t$ seconds.

$$S = t^2 \text{ at } A$$

and at B

$$s + \delta s = (t + \delta t)^2$$

$$\therefore \frac{\delta s}{\delta t} = \frac{(t + \delta t)^2 - t^2}{\delta t} = \text{av. vel. during } \delta t \text{ of a sec.}$$

To get the true velocity at the end of $t$ seconds we must

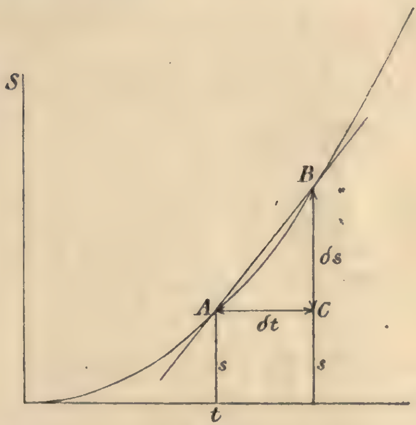


FIG. 19.

assume $\delta t$ to be indefinitely small, and consequently $\delta s$ will become indefinitely small. When $\delta t$ is indefinitely small, that is smaller than ·000 000 000 1 it is denoted by $dt$ and the corresponding increment of space by $ds$

$$\therefore \frac{ds}{dt} = \frac{(t + dt)^2 - t^2}{dt} = \frac{2tdt + (dt)^2}{dt} = 2t + dt$$

$$\therefore \frac{ds}{dt} = 2t = \text{the velocity at the end of } t \text{ secs.}$$

since $dt$ may be neglected.

When $dt$ is indefinitely small A and B must be indefinitely near to each other and consequently the straight line joining A and B must be a tangent to the curve. $ds$ and $dt$ may be indefinitely small but the ratio of $ds$ to $dt$ may be very large or very small depending upon the slope of the curve.

$\frac{ds}{dt}$ means the rate of change of $s$ with regard to $t$.

It means the slope of the curve or tangent of its direction at any point.

It is also called the Differential coefficient of $s$ with regard to $t$

We now proceed to find the Differential coefficient of $x^n$ where $n$ is positive, negative, integral or fractional.

$$\text{Let } y = x^n \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $x$ be increased by a finite increment $\delta x$ and in consequence let $y$ become $y + \delta y$

$$\therefore y + \delta y = (x + \delta x)^n \quad . \quad . \quad . \quad (2)$$

Subtract (1) from (2) and we have

$$\delta y = (x + \delta x)^n - x^n$$

and on dividing by $\delta x$ we have

$$\frac{\delta y}{\delta x} = \frac{(x + \delta x)^n - x^n}{\delta x} \quad . \quad . \quad . \quad . \quad (3)$$

The limiting value of this expression when $\delta x$ is indefinitely small is called the Differential coefficient of $y$ with regard to $x$ On expanding $(x + \delta x)^n$ in (3) by the Binomial Theorem, we get

$$\frac{\delta y}{\delta x} = \frac{x^n + nx^{n-1}\delta x + \frac{n(n-1)}{\underline{|2}}x^{n-2}(\delta x)^2 \ldots - x^n}{\delta x}$$

$$\therefore \frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)x^{n-2}}{\underline{|2}}\delta x \ldots$$

If $\delta x$ be indefinitely small, then we have in the limit

$$\frac{dy}{dx} = nx^{n-1} \; . \; . \; . \; . \; . \; . \quad (4)$$

The expansion of $(x + \delta x)^n$ is true for all real values of $n$, therefore (4) is true for all real values of $n$

If $y = x^9$ then $\frac{dy}{dx} = 9x^8$ if $y = x^{\frac{1}{7}}$ then $\frac{dy}{dx} = \frac{1}{7}x^{-\frac{6}{7}}$

if $y = x^{-\frac{5}{3}}$ then $\frac{dy}{dx} = -\frac{5}{3}x^{-\frac{8}{3}}$

**In order to differentiate a power of $x$ multiply by the index and diminish the index by unity.** And we must not forget that we are finding the rate of increase of the power of $x$ with regard to $x$

If $\quad y = x^2$

then $\quad \frac{\delta y}{\delta x} = \frac{(x + \delta x)^2 - x^2}{\delta x}$

Let $\quad x = 10$ and $\delta x = \cdot 1$

$$\therefore \frac{\delta y}{\delta x} = \frac{(10 \cdot 1)^2 - 10^2}{\cdot 1} = 20 \cdot 1$$

This gives the approximate value of the rate of increase of $y$ with regard to $x$ when $x = 10$, the true value when $x = 10$ is given by

$$\frac{dy}{dx} = 2x = 20$$

**The differential coefficient of an algebraic sum of any number of functions of $x$ is the algebraic sum of the differential coefficients of the functions.**

Let $\quad y = u + v - z \, , \; . \; . \; . \; . \; . \quad (1)$

where $u$, $v$ and $z$ are functions of $x$ and let $x$ be increased to $x + \delta x$ and in consequence let

$$\begin{array}{lll} y & \text{become} & y+\delta y \\ u & ,, & u+\delta u \\ v & ,, & v+\delta v \\ z & ,, & z+\delta z \end{array}$$

then $\quad y+\delta y \;=\; u+\delta u+v+\delta v-(z+\delta z)$. . (2)

and by subtracting (1) from (2) we have

$$\delta y = \delta u+\delta v-\delta z$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\delta u}{\delta x}+\frac{\delta u}{\delta x}-\frac{\delta z}{\delta x}$$

and in the limit when $\delta x$ is diminished indefinitely we have

$$\frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx}-\frac{dz}{dx} \quad . \quad . \quad . \quad . \quad (3)$$

which proves the rule that is true for any number of functions.

*Example.*

$$y = x^6-7x^5+3x^4+2x^2+4$$

$$\frac{dy}{dx} = 6x^5-35x^4+12x^3+4x$$

**Differential coefficient of a product of two or more functions.**

Let $y = uv$ where $u$ and $v$ are functions of $x$ and let $x$ be increased to $x+\delta x$ and in consequence let

$$\begin{array}{lll} y & \text{become} & y+\delta y \\ u & ,, & u+\delta u \\ v & ,, & v+\delta v \end{array}$$

$$\therefore y+\delta y = (u+\delta u)(v+\delta v)$$

$$y = uv$$

$$\therefore \delta y = v\delta u+u\delta v+\delta u\delta v$$

$\delta u\delta v$ may be neglected since it is indefinitely small compared with $\delta u$ or $\delta v$ when $\delta x$ is diminished indefinitely

$$\therefore \frac{dy}{dx} = v\frac{du}{dx}+u\frac{dv}{dx}$$

**The differential coefficient of a product of two functions of $x$ is obtained by multiplying each function by the differential coefficient of the other function, and taking the sum of the products.**

Similarly, if $y = uvwz$ where $u$, $v$, $w$ and $z$ are functions of $x$, we have

$$\frac{dy}{dx} = vwz\frac{du}{dx} + uwz\frac{dv}{dx} + uvz\frac{dw}{dx} + uvw\frac{dz}{dx}$$

The general proof of this is deferred until we differentiate $\log x$

**Differential Coefficient of a Quotient.**

Let
$$y = \frac{u}{v}$$

where $u$ and $v$ are functions of $x$

On multiplying by $v$ we have

$$u = vy$$

therefore by the product rule, we have

$$\frac{du}{dx} = v\frac{dy}{dx} + y\frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{v}\cdot\frac{du}{dx} - \frac{y}{v}\cdot\frac{dv}{dx} = \frac{1}{v}\cdot\frac{du}{dx} - \frac{u}{v^2}\cdot\frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

**The differential coefficient of a quotient is obtained by differentiating the numerator and multiplying by the denominator, then differentiate the denominator and multiply by the numerator and subtract the latter product from the former and divide the difference by the square of the denominator.**

*Example.*
$$y = \frac{x^2+2}{x^4+4}$$
$$\frac{dy}{dx} = \frac{2x(x^4+4) - 4x^3(x^2+2)}{(x^4+4)^2}$$
$$\therefore \frac{dy}{dx} = \frac{-2(x^5+4x^3-4x)}{(x^4+4)^2}$$

**Differentation of a function of a function.**

Let $y = f(z)$ where $z = \phi(x)$ $\therefore y = f\{\phi(x)\}$

It is required to find
$$\frac{dy}{dx}$$

We have
$$\frac{y_1 - y}{x_1 - x} = \frac{y_1 - y}{z_1 - z} \times \frac{z_1 - z}{x_1 - x}$$

for finite differences of $x$, $y$, and $z$, and assuming that this relation holds good however small the differences may be, we have in the limit
$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

Now
$$\frac{dy}{dz} = f'(z) \quad \text{and} \quad \frac{dz}{dx} = \phi'(x)$$

where $f'(z)$ and $\phi'(x)$ stand for the first derived functions of $f(z)$ and $\phi(x)$ respectively
$$\therefore \frac{dy}{dx} = f'(z)\phi'(x) = f'\{\phi(x)\}\phi'(x)$$

An example will make this clear.

Let $y = (a^3 + x^3)^4$

Assume $z = a^3 + x^3$
$$\therefore y = z^4 \quad \text{and} \quad \frac{dy}{dz} = 4z^3$$

also
$$\frac{dz}{dx} = 3x^2$$
$$\therefore \frac{dy}{dx} = 4z^3 \times 3x^2 = 4(a^3+x^3)^3 \times 3x^2$$
$$= 12x^2(a^3+x^3)^3$$

*Example.*—Let $y = (ax^2 + bx + c)^{\frac{3}{4}}$

Let $z = ax^2 + bx + c \quad \therefore \frac{dz}{dx} = 2ax + b$

and $y = z^{\frac{3}{4}}$

$$\therefore \frac{dy}{dz} = \frac{3}{4}z^{-\frac{1}{4}} = \frac{3}{4z^{\frac{1}{4}}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{4z^{\frac{1}{4}}} \times (2ax + b)$$

$$= \frac{3(2ax + b)}{4(ax^2 + bx + c)^{\frac{1}{4}}}$$

Differentiation of $\log_a x$

Let $y = \log_a x$

we have $dy = \log_a (x + dx) - \log_a x = \log_a \left(1 + \frac{dx}{x}\right)$

therefore $\frac{dy}{dx} = \frac{1}{dx} \log_a \left(1 + \frac{dx}{x}\right)$. . . . . . ($a$)

Let $dx = \alpha x$ where $\alpha$ is indefinitely small and ($a$) becomes

$$\frac{dy}{dx} = \frac{1}{\alpha x} \log_a (1 + \alpha) = \frac{1}{x} \log_a (1 + \alpha)^{\frac{1}{\alpha}}$$

Expanding $(1 + \alpha)^{\frac{1}{\alpha}}$ by the Binomial Theorem, we have

$$(1 + \alpha)^{\frac{1}{\alpha}}$$

$$= 1 + \frac{1}{\alpha} \times \alpha + \frac{\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)}{\lfloor 2} \alpha^2 + \frac{\frac{1}{\alpha}\left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 2\right)}{\lfloor 3} \alpha^3 + \text{etc.}$$

$$= 1 + 1 + \frac{1 - \alpha}{\lfloor 2} + \frac{(1 - \alpha)(1 - 2\alpha)}{\lfloor 3} + \text{etc.}$$

and since $\alpha$ is indefinitely small we may neglect it, therefore

$$(1+\alpha)^{\frac{1}{\alpha}} = 1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 5}+\text{etc.}$$

$=$ $e$ the base of the Napierian logarithms, the numerical value of which is 2·71828 . . . .

Hence $$\frac{dy}{dx} = \frac{1}{x}\log_a e$$

If $y = \log_a x$ then $x = a^y$

But $a^y = 1 + y\log a + \frac{y^2}{\lfloor 2}(\log a)^2 + \frac{y^3}{\lfloor 3}(\log a)^3$ etc.

$$\therefore x = 1 + y\log a + \frac{y^2}{\lfloor 2}(\log a)^2 + \frac{y^3}{\lfloor 3}(\log a)^3 + \text{etc.}$$

$$\therefore \frac{dx}{dy} = \log a + y(\log a)^2 + \frac{y^2}{\lfloor 2}(\log a)^3 + \text{etc.}$$

$$= \log a\{1 + y\log a + \frac{y^2}{\lfloor 2}(\log a)^2 + \text{etc.}\}$$

$$= x\log a$$

$$\therefore \frac{dy}{dx} = \frac{1}{x\log a} = \frac{1}{x}\log_a e = \frac{1}{x} \text{ when } a = e.$$

To differentiate $\log y$ with regard to $x$

Let $z = \log y \quad \therefore dz = \frac{1}{y}dy$

$$\therefore \frac{dz}{dx} = \frac{1}{y}\cdot\frac{dy}{dx}$$

Let $y = uvwz$ where $u$ $v$ $w$ and $z$ are functions of Taking the logarithm of both sides, we have

$$\log y = \log u + \log v + \log w + \log z$$

On differentiating both sides with regard to $x$ we have

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} + \frac{1}{w}\cdot\frac{dw}{dx} + \frac{1}{z}\cdot\frac{dz}{dx}$$

$$\therefore \frac{dy}{dx} = vwz\frac{du}{dx} + uwz\frac{dv}{dx} + uvz\frac{dw}{dx} + uvw\frac{dz}{dx}$$

It is important to bear in mind that

$$\frac{\sin x}{x} = 1$$

where $x$ is an indefinitely small angle.

$$\text{Sin } x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \text{etc.}$$

$$\therefore \frac{\sin x}{x} = 1 - \frac{x^2}{\underline{|3}} + \frac{x^4}{\underline{|5}} - \text{etc.}$$

In this expression if $x$ be indefinitely small all the terms on the right-hand vanish except the first

$$\therefore \frac{\sin x}{x} = 1$$

where $x$ is indefinitely small.

Let AC be a very small arc of a circle of radius unity, and let AD be the perpendicular from A on OC then it

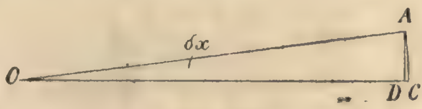


FIG. 20.

is obvious that if the angle $\delta x$ be very small AD is ultimately equal to AC, but

$$\frac{AC}{AO} = \delta x \text{ in radians, and } \frac{AD}{AO} = \sin \delta x$$

$$\therefore \frac{\sin \delta x}{\delta x} = \frac{AD}{AO} \div \frac{AC}{AO} = \frac{AD}{AC} = 1$$

when $\delta x$ is indefinitely small.

**Differentiation of the Trigonometrical functions.**

Let $y = \sin x \quad \therefore y + \delta y = \sin (x + \delta x)$ and by subtraction we have

$$\delta y = \sin(x + \delta x) - \sin x$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\sin(x+\delta x) - \sin x}{\delta x} = 2\cos\left(x + \frac{\delta x}{2}\right)\frac{\sin\frac{\delta x}{2}}{\delta x}$$

$$= \cos\left(x + \frac{\delta x}{2}\right)\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$\therefore \frac{dy}{dx} = \cos(x + 0)(1) = \cos x$$

Since $\frac{\sin\theta}{\theta} = 1$ when $\theta$ is indefinitely small

therefore $\frac{dy}{dx} = \cos x$ if $y = \sin x$

Let $y = \cos x$, $\therefore y + \delta y = \cos(x + \delta x)$

and $\delta y = \cos(x + \delta x) - \cos x = -2\sin\left(x + \frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$

$$\therefore \frac{\delta y}{\delta x} = -\sin\left(x + \frac{\delta x}{2}\right)\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$\therefore \frac{dy}{dx} = -\sin x$$

when $\delta x$ is diminished indefinitely.

Let $y = \tan x$ $\therefore y + \delta y = \tan(x + \delta x)$

$$\therefore \delta y = \tan(x + \delta x) - \tan x = \frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x + \delta x)\cos x - \cos(x + \delta x)\sin x}{\cos(x + \delta x)\cos x}$$

$$= \frac{\sin\delta x}{\cos(x + \delta x)\cos x}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\dfrac{\sin \delta x}{\delta x}}{\cos (x + \delta x) \cos x}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos^2 x} = \sec^2 x$$

Let $y = \cot x \quad \therefore y + \delta y = \cot (x + \delta x)$

and $\delta y = \cot (x + \delta x) - \cot x = \dfrac{\cos (x + \delta x)}{\sin (x + \delta x)} - \dfrac{\cos x}{\sin x}$

$$\therefore \frac{\delta y}{\delta x} = \frac{\dfrac{\cos (x + \delta x) \sin x - \sin (x + \delta x) \cos x}{\delta x}}{\sin (x + \delta x) \sin x}$$

$$= -\frac{\dfrac{\sin \delta x}{\delta x}}{\sin (x + \delta x) \sin x}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sin^2 x} = -\operatorname{cosec}^2 x$$

Let $y = \sec x \quad \therefore y + \delta y = \sec (x + \delta x)$

and $\delta y = \sec (x + \delta x) - \sec x = \dfrac{1}{\cos (x + \delta x)} - \dfrac{1}{\cos x}$

$$= \frac{\cos x - \cos (x + \delta x)}{\cos (x + \delta x) \cos x} = \frac{2 \sin \left(x + \dfrac{\delta x}{2}\right) \sin \dfrac{\delta x}{2}}{\cos (x + \delta x) \cos x}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{\sin \left(x + \dfrac{\delta x}{2}\right)}{\cos (x + \delta x) \cos x} \left( \frac{\sin \dfrac{\delta x}{2}}{\dfrac{\delta x}{2}} \right)$$

$$\therefore \frac{dy}{dx} = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

Let $y = \operatorname{cosec} x \quad \therefore y + \delta y = \operatorname{cosec} (x + \delta x)$

and $\delta y = \operatorname{cosec} (x + \delta x) - \operatorname{cosec} x = \dfrac{1}{\sin (x + \delta x)} - \dfrac{1}{\sin x}$

$$= \frac{\sin x - \sin (x + \delta x)}{\sin (x + \delta x) \sin x} = \frac{-2 \cos \left(x + \dfrac{\delta x}{2}\right) \sin \dfrac{\delta x}{2}}{\sin (x + \delta x) \sin x}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{-\cos\left(x+\frac{\delta x}{2}\right)}{\sin(x+\delta x)\sin x}\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$\therefore \frac{dy}{dx} = \frac{-\cos x}{\sin^2 x} = -\operatorname{cosec} x \cot x$$

$$\text{If } y = \sin x \quad \text{then } \frac{dy}{dx} = \cos x$$

$$\text{,, } y = \cos x \quad \text{,, } \frac{dy}{dx} = -\sin x$$

$$\text{,, } y = \tan x \quad \text{,, } \frac{dy}{dx} = \sec^2 x$$

$$\text{,, } y = \cot x \quad \text{,, } \frac{dy}{dx} = -\operatorname{cosec}^2 x$$

$$\text{,, } y = \sec x \quad \text{,, } \frac{dy}{dx} = \sec x \tan x$$

$$\text{,, } y = \operatorname{cosec} x \quad \text{,, } \frac{dy}{dx} = -\operatorname{cosec} x \cot x$$

**Differentiation of the inverse Trigonometrical functions.**

Let $y = \sin^{-1} x$

$$\therefore x = \sin y$$

and $$\frac{dx}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Let $y = \cos^{-1} x \quad \therefore x = \cos y$

and we have

$$\frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Let $y = \tan^{-1} x \qquad \therefore x = \tan y$

$$\therefore \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + x^2}$$

Let $y = \cot^{-1} x \qquad \therefore x = \cot y$

$$\therefore \frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1 + \cot^2 y) = -(1 + x^2)$$

$$\therefore \frac{dy}{dx} = -\frac{1}{1 + x^2}$$

Let $y = \sec^{-1} x \qquad \therefore x = \sec y$

$$\therefore \frac{dx}{dy} = \sec y \tan y = \sec y\sqrt{\sec^2 y - 1}$$

$$= x\sqrt{x^2 - 1}$$

$$\therefore \frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

Let $y = \operatorname{cosec}^{-1} x \qquad \therefore x = \operatorname{cosec} y$

$$\therefore \frac{dx}{dy} = -\operatorname{cosec} y \cot y = -\operatorname{cosec} y\sqrt{\operatorname{cosec}^2 y - 1}$$

$$= -x\sqrt{x^2 - 1}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\left.\begin{array}{llll}
\text{If } y = \sin^{-1} x & \text{then} & \dfrac{dy}{dx} = & \dfrac{1}{\sqrt{1 - x^2}} \\
\text{,, } y = \cos^{-1} x & \text{,,} & \dfrac{dy}{dx} = & -\dfrac{1}{\sqrt{1 - x^2}}
\end{array}\right\}$$

$$\left.\begin{array}{llll}
\text{,, } y = \tan^{-1} x & \text{,,} & \dfrac{dy}{dx} = & \dfrac{1}{1 + x^2} \\
\text{,, } y = \cot^{-1} x & \text{,,} & \dfrac{dy}{dx} = & -\dfrac{1}{1 + x^2}
\end{array}\right\}$$

$$\left.\begin{array}{llll}
\text{,, } y = \sec^{-1} x & \text{,,} & \dfrac{dy}{dx} = & \dfrac{1}{x\sqrt{x^2 - 1}} \\
\text{,, } y = \operatorname{cosec}^{-1} x & \text{,,} & \dfrac{dy}{dx} = & -\dfrac{1}{x\sqrt{x^2 - 1}}
\end{array}\right\}$$

Differentiation of $a^x$ and $e^{ax}$

Let $y = a^x \quad \therefore \log y = x \log a$

$$\therefore \frac{1}{y} \cdot \frac{dy}{dx} = \log a$$

$$\therefore \frac{dy}{dx} = y \log a = a^x \log a$$

Let $y = e^{ax} \quad \therefore \log y = ax$

$$\therefore \frac{1}{y} \frac{dy}{dx} = a \qquad \therefore \frac{dy}{dx} = ay = ae^{ax}$$

**Differentiation of the Hyperbolic functions.**

Let $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$

$$\therefore \frac{dy}{dx} = \frac{e^x + e^{-x}}{2} = \cosh x$$

Let $y = \cosh x = \dfrac{e^x + e^{-x}}{2}$

$$\therefore \frac{dy}{dx} = \frac{e^x - e^{-x}}{2} = \sinh x$$

Let $y = \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\frac{dy}{dx} = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = \frac{4}{(e^x + e^{-x})^2}$$
$$= \operatorname{sech}^2 x$$

Let $y = \coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$$\therefore \frac{dy}{dx} = \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{(e^x - e^{-x})^2} = \frac{-4}{(e^x - e^{-x})^2}$$
$$= -\operatorname{cosech}^2 x$$

Let $y = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}} = 2(e^x + e^{-x})^{-1}$

$$\therefore \frac{dy}{dx} = -2(e^x + e^{-x})^{-2}(e^x - e^{-x})$$

$$= -\frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2} = -\operatorname{sech} x \tanh x$$

$$\text{Let } y = \operatorname{cosech} x = \frac{2}{e^x - e^{-x}} = 2(e^x - e^{-x})^{-1}$$

$$\therefore \frac{dy}{dx} = -2(e^x - e^{-x})^{-2}(e^x + e^{-x})$$

$$= -\frac{2(e^x + e^{-x})}{(e^x - e^{-x})^2} = -\operatorname{cosech} x \coth x$$

$$\text{If } y = \sinh x \quad \text{then} \quad \frac{dy}{dx} = \cosh x$$

$$\text{,, } y = \cosh x \quad \text{,,} \quad \frac{dy}{dx} = \sinh x$$

$$\text{,, } y = \tanh x. \quad \text{,,} \quad \frac{dy}{dx} = \operatorname{sech}^2 x$$

$$\text{,, } y = \coth x \quad \text{,,} \quad \frac{dy}{dx} = -\operatorname{cosech}^2 x$$

$$\text{,, } y = \operatorname{sech} x \quad \text{,,} \quad \frac{dy}{dx} = -\operatorname{sech} x \tanh x$$

$$\text{,, } y = \operatorname{cosech} x \quad \text{,,} \quad \frac{dy}{dx} = -\operatorname{cosech} x \coth x$$

**Differentiation of the inverse Hyperbolic functions.**

$$\text{Let } y = \sinh^{-1} x \qquad \therefore x = \sinh y$$

$$\frac{dx}{dy} = \cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$$

$$\text{Let } y = \cosh^{-1} x \qquad \therefore x = \cosh y$$

$$\therefore \frac{dx}{dy} = \sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}} \quad \text{where } x^2 > 1$$

Let $y = \tanh^{-1} x \quad \therefore x = \tanh y$

and $\dfrac{dx}{dy} = \operatorname{sech}^2 y = 1 - \tanh^2 y = 1 - x^2$

$$\therefore \frac{dy}{dx} = \frac{1}{1 - x^2} \quad \text{where} \quad x^2 < 1$$

Let $y = \coth^{-1} x \quad \therefore x = \coth y$

$\dfrac{dx}{dy} = -\operatorname{cosech}^2 y = -(\coth^2 y - 1) = -(x^2 - 1)$

$$\therefore \frac{dy}{dx} = \frac{1}{1 - x^2} \quad \text{where} \quad x^2 > 1$$

Let $y = \operatorname{sech}^{-1} x \quad \therefore x = \operatorname{sech} y$

$$\frac{dx}{dy} = -\operatorname{sech} y \tanh y = -x\sqrt{1 - x^2}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x\sqrt{1 - x^2}} \quad \text{where} \quad x^2 < 1$$

Let $y = \operatorname{cosech}^{-1} x \quad \therefore x = \operatorname{cosech} y$

$$\frac{dx}{dy} = -\operatorname{cosech} y \coth y = -x\sqrt{x^2 + 1}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{x\sqrt{x^2 + 1}}$$

Differentiation of $\log f(x)$

Let $y = \log f(x) = \log z \quad \text{where } z = f(x)$

we have $\dfrac{dy}{dz} = \dfrac{1}{z}$ and $\dfrac{dz}{dx} = f'(x)$

$$\therefore \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

The differential coefficient of the logarithm of a function of $x$ may be expressed in the form of a fraction the numerator of which is the function of $x$ differentiated and the denominator is the function of $x$

*Example.*
$$y = \log(x^3 + 2x^2 + 4)$$
$$y = \log z \quad \text{where } z = x^3 + 2x^2 + 4$$
$$\frac{dy}{dz} = \frac{1}{z} \quad \text{and} \quad \frac{dz}{dx} = 3x^2 + 4x$$
$$\therefore \frac{dy}{dx} = \frac{3x^2 + 4x}{x^3 + 2x^2 + 4}$$

*Example.*
$$y = \log(x^2 + 3x + 1)^3 = 3\log(x^2 + 3x + 1)$$
$$y = 3\log z \quad \text{where} \quad z = x^2 + 3x + 1$$
$$\frac{dy}{dz} = \frac{3}{z} \quad \text{and} \quad \frac{dz}{dx} = 2x + 3$$
$$\therefore \frac{dy}{dx} = \frac{3(2x + 3)}{x^2 + 3x + 1}$$

*Example.*
$$y = \log(x + \sqrt{a^2 + x^2})$$
$$y = \log z \quad \text{where} \quad z = x + \sqrt{a^2 + x^2}$$
$$\frac{dy}{dz} = \frac{1}{z} \quad \text{and} \quad \frac{dz}{dx} = 1 + \frac{x}{\sqrt{a^2 + x^2}}$$
$$\therefore \frac{dy}{dx} = \frac{x + \sqrt{a^2 + x^2}}{\sqrt{a^2 + x^2}} \times \frac{1}{x + \sqrt{a^2 + x^2}} = \frac{1}{\sqrt{a^2 + x^2}}$$

*Example.*
$$y = \frac{\sqrt{x^2 + a^2}\sqrt[3]{x^2 + b^2}}{\sqrt[5]{x^2 + e^2}\sqrt[7]{x^2 + d^2}}$$
$$\therefore \log y = \tfrac{1}{2}\log(x^2 + a^2) + \tfrac{1}{3}\log(x^2 + b^2) - \tfrac{1}{5}\log(x^2 + e^2) - \tfrac{1}{7}\log(x^2 + d^2)$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{x}{x^2 + a^2} + \frac{2x}{3(x^2 + b^2)} - \frac{2}{5} \times \frac{x}{x^2 + e^2} - \frac{2x}{7(x^2 + d^2)}$$

*Example.*
$$y = \tan^{-1}\frac{\sqrt{x} + \sqrt{a} + \sqrt{b} - \sqrt{abx}}{1 - \sqrt{ax} - \sqrt{bx} - \sqrt{ab}}$$
$$\therefore y = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{a} + \tan^{-1}\sqrt{b}$$
$$\therefore \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \times \frac{1}{1 + x}$$

*Example.*

$$y = e^{ax} \sin bx$$

$$\frac{dy}{dx} = ae^{ax} \sin bx + e^{ax} b \cos bx$$

$$= e^{ax}(a \sin bx + b \cos bx)$$

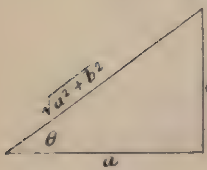


FIG. 21.

Let $a$ be the base of a right-angled triangle and $b$ the vertical,

then $\quad a = \sqrt{a^2 + b^2} \cos \theta$

and $\quad b = \sqrt{a^2 + b^2} \sin \theta$

$$\therefore e^{ax}(a \sin bx + b \cos bx)$$

$$= \sqrt{a^2 + b^2}\,(\sin bx \cos \theta + \cos bx \sin \theta)$$

$$= \sqrt{a^2 + b^2} \sin (bx + \theta)$$

*Example.*

$$y = \sin a^{\frac{1}{x}}$$

Let $\quad z = a^{\frac{1}{x}}$

$$\therefore y = \sin z$$

$$\frac{dy}{dz} = \cos z$$

also $\quad \log z = \frac{1}{x} \log a = x^{-1} \log a$

$$\frac{1}{z} \cdot \frac{dz}{dx} = -x^{-2} \log a = -\frac{\log a}{x^2}$$

$$\therefore \frac{dz}{dx} = -z \frac{\log a}{x^2} = -\frac{a^{\frac{1}{x}}}{x^2} \log a$$

$$\therefore \frac{dy}{dx} = -\frac{a^{\frac{1}{x}}}{x^2} \log a \times \cos a^{\frac{1}{x}}$$

Suppose the displacement of a body from a fixed point be given by the equation

$$s = 16t^2 + 12t + 4$$

where $s$ is the displacement in feet, and $t$ is the time in seconds reckoned from the instant the body was 4 feet from the starting point. If we differentiate $s$ with regard to $t$ we get

$$\frac{ds}{dt} = 32t + 12$$

But $\frac{ds}{dt}$ means the rate of change of displacement with regard to time $t$, and is therefore the velocity of the body at any time $t$

$$\therefore v = \frac{ds}{dt} = 32t + 12$$

when $t = 0$ the velocity $v = 12$ feet per second
„ $t = 1$ „ „ $v = 44$ „ „ „

Again suppose $s = a + b \sin \theta + c \sin 2\theta$ where $s$ represents the displacement of a body for any angle, $\theta$ The velocity is obtained by differentiating $s$ with regard to time $t$ By taking the differentials of all the terms we have

$$ds = b \cos \theta d\theta + 2c \cos 2\theta d\theta$$

and on dividing by $dt$ we have

$$\frac{ds}{dt} = b \cos \theta \frac{d\theta}{dt} + 2c \cos 2\theta \frac{d\theta}{dt}$$

Now $\frac{d\theta}{dt}$ means the rate of change of angle with regard to time and is therefore angular velocity, therefore the linear velocity of the body is given by

$$v = \frac{ds}{dt} = (b \cos \theta + 2c \cos 2\theta)\frac{d\theta}{dt}$$

Suppose $\theta$ to increase from 0 to $2\pi$ in 4 seconds then $\frac{d\theta}{dt} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\therefore v = (b \cos \theta + 2c \cos 2\theta)\frac{\pi}{2}$$

A point P moves on the circumference of a circle with a uniform velocity, making one complete turn in 2 seconds, find the velocity of the foot of the perpendicular from P on a diameter, the radius being 2 feet, when $\theta = 60^\circ$

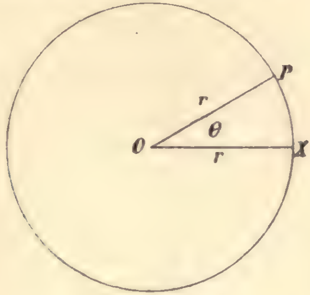


FIG. 22.

Let the displacement of the foot of the perpendicular from its mean position be $s$, then $s = 2\cos\theta$.

and $$v = \frac{ds}{dt} = -2\sin\theta\frac{d\theta}{dt}$$

Here $$\frac{d\theta}{dt} = \frac{2\pi}{2} = \pi \text{ and } \sin\theta = \frac{\sqrt{3}}{2}$$

$$\therefore v = -2 \times \frac{\sqrt{3}}{2} \times \pi = -\sqrt{3} \times 3{\cdot}1416 \text{ ft. per sec.}$$

The negative sign indicates that $s$ is diminishing as $\theta$ increases.

## LIST OF DIFFERENTIAL COEFFICIENTS

$$y = x^n \qquad \frac{dy}{dx} = nx^{n-1}$$

$$y = \log_a x \qquad \frac{dy}{dx} = \frac{\log_a e}{x}$$

$$y = a^x \qquad \frac{dy}{dx} = a^x \log a$$

$$y = \sin ax \qquad \frac{dy}{dx} = a \cos ax$$

$$y = \cos ax \qquad \frac{dy}{dx} = -a \sin ax$$

$$y = \tan ax \qquad \frac{dy}{dx} = a \sec^2 ax$$

$$y = \cot ax \qquad \frac{dy}{dx} = -a \operatorname{cosec}^2 ax$$

$$y = \sec ax \qquad \frac{dy}{dx} = a \sec ax \tan ax$$

$$y = \operatorname{cosec} ax \qquad \frac{dy}{dx} = -a \operatorname{cosec} ax \cot ax$$

$$y = \sin^{-1} \frac{x}{a} \qquad \frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}$$

$$y = \cos^{-1} \frac{x}{a} \qquad \frac{dy}{dx} = -\frac{1}{\sqrt{a^2 - x^2}}$$

$$y = \tan^{-1} \frac{x}{a} \qquad \frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

$$y = \cot^{-1} \frac{x}{a} \qquad \frac{dy}{dx} = -\frac{a}{a^2 + x^2}$$

$$y = \sec^{-1} \frac{x}{a} \qquad \frac{dy}{dx} = \frac{a}{x\sqrt{x^2 - a^2}}$$

$$y = \operatorname{cosec}^{-1} \frac{x}{a} \qquad \frac{dy}{dx} = -\frac{a}{x\sqrt{x^2 - a^2}}$$

$$y = \operatorname{versin}^{-1} \frac{x}{a} \qquad \frac{dy}{dx} = \frac{a}{\sqrt{2ax - x^2}}$$

$$y = \operatorname{coversin}^{-1} \frac{x}{a} \qquad \frac{dy}{dx} = -\frac{a}{\sqrt{2ax - x^2}}$$

$$y = \log (x + \sqrt{a^2 + x^2}) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{a^2 + x^2}}$$

### *Examples on differentiation*

1. $y = x^3 \qquad \dfrac{dy}{dx} = 3x^2$

2. $y = 6x^{\frac{1}{3}} \qquad \dfrac{dy}{dx} = 2x^{-\frac{2}{3}}$

3. $y = ax^{-\frac{3}{4}} \qquad \dfrac{dy}{dx} = -\dfrac{3a}{4}x^{-\frac{7}{4}}$

4. $y = \sqrt[3]{(x^5)} \qquad \dfrac{dy}{dx} = \dfrac{5}{3}x^{\frac{2}{3}}$

5. $y = x^2 - 3x + 2 \qquad \dfrac{dy}{dx} = 2x - 3$

6. $y = x^5 - x^4 - 3x - 1 \qquad \dfrac{dy}{dx} = 5x^4 - 4x^3 - 3$

7. $y = x(x-1)(x-2) \qquad \dfrac{dy}{dx} = 3x^2 - 6x + 2$

8. $y = \dfrac{x+a}{x+b} \qquad \dfrac{dy}{dx} = \dfrac{b-a}{(x+b)^2}$

9. $y = (x^3 - a^3)(b + x) \qquad \dfrac{dy}{dx} = 4x^3 + 3bx^2 - a^3$

10. $y = (x+a)^n(x+b)^m$

$$\frac{dy}{dx} = (x+a)^{n-1}(x+b)^{m-1}\{(m+n)x + bn + am\}$$

11. $y = \sqrt{\dfrac{1+x^2}{1-x^2}} \qquad \dfrac{dy}{dx} = \dfrac{2x}{(1-x^2)^{\frac{3}{2}}\sqrt{(1+x^2)}}$

12. $y = \sin^2 x \qquad \dfrac{dy}{dx} = \sin 2x$

13. $y = \sin x^2 \qquad \dfrac{dy}{dx} = 2x \cos x^2$

14. $y = \tan nx \qquad \dfrac{dy}{dx} = n \sec^2 nx$

15. $y = \tan x^n \qquad \dfrac{dy}{dx} = nx^{n-1}\sec^2 x^n$

16. $y = \sec^{-1} x^3 \qquad \dfrac{dy}{dx} = \dfrac{3}{x\sqrt{(x^6 - 1)}}$

17. $y = \cos^{-1}(3x - 1) \qquad \dfrac{dy}{dx} = -\dfrac{\sqrt{3}}{\sqrt{x(2 - 3x)}}$

18. $y = a\sqrt[3]{x} \qquad \dfrac{dy}{dx} = \dfrac{a}{3\sqrt[3]{x^2}}$

19. $y = x^4 - 3x^2 + 7 \qquad \dfrac{dy}{dx} = 4x^3 - 6x$

20. $y = \dfrac{1 - x}{1 + x^2} \qquad \dfrac{dy}{dx} = \dfrac{x^2 - 2x - 1}{(1 + x^2)^2}$

21. $y = \sqrt{(a^2 - x^2)} \qquad \dfrac{dy}{dx} = -\dfrac{x}{\sqrt{a^2 - x^2}}$

22. $y = (x + 2)(x^2 - 4) \qquad \dfrac{dy}{dx} = 3x^2 + 4x - 4$

23. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \dfrac{dy}{dx} = -\dfrac{4}{(e^x - e^{-x})^2}$

24. $y = \log(e^x - e^{-x}) \qquad \dfrac{dy}{dx} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

25. $y = \dfrac{c}{x^3} \qquad \dfrac{dy}{dx} = -\dfrac{3c}{x^4}$

26. $y = (1 + kx^3)^{\frac{1}{n}} \qquad \dfrac{dy}{dx} = \dfrac{3kx^2}{n}(1 + kx^3)^{\frac{1-n}{n}}$

27. $y = \log ax^2 \qquad \dfrac{dy}{dx} = \dfrac{2}{x}$

28. $y = \log \sin x \qquad \dfrac{dy}{dx} = \cot x$

29. $y = a^x(1 + \log x)$

$$\frac{dy}{dx} = a^x\left\{(1 + \log x)\log a + \frac{1}{x}\right\}$$

30. $y = x^x e^x \qquad \dfrac{dy}{dx} = x^x e^x(2 + \log x)$

31. $y = \log(\sin^{-1} x) \qquad \dfrac{dy}{dx} = \dfrac{1}{\sin^{-1} x\sqrt{(1 - x^2)}}$

32. $y = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} \qquad \dfrac{dy}{dx} = \dfrac{1}{x^2 + a^2}$

33. $y = \dfrac{1}{a}\log\dfrac{x}{a + \sqrt{a^2 \pm x^2}}$

$$\frac{dy}{dx} = \frac{1}{x\sqrt{a^2 \pm x^2}}$$

34. $y = \sec a^x \qquad \dfrac{dy}{dx} = a^x \log a \sec a^x \tan a^x$

35. $y = \left(\dfrac{1}{x}\right)^x \qquad \dfrac{dy}{dx} = \left(\dfrac{1}{x}\right)^x\left(\log\dfrac{1}{x} - 1\right)$

36. $y = \sin x \sin^{-1} x \qquad \dfrac{dy}{dx} = \dfrac{\sin x}{\sqrt{(1 - x^2)}} + \sin^{-1} x \cos x$

37. $y = (p + qx)e^{ax} \qquad \dfrac{dy}{dx} = e^{ax}\{q + a(p + qx)\}$

38. $y = \sin^{-1}\sqrt{1 - x^2} \qquad \dfrac{dy}{dx} = -\dfrac{1}{\sqrt{1 - x^2}}$

39. $y = \tan^{-1}\dfrac{3x}{1 - x^2} \qquad \dfrac{dy}{dx} = \dfrac{3(1 + x^2)}{1 + 7x^2 + x^4}$

40. $y = \log\sqrt{\left(\dfrac{1 + x}{1 - x}\right)} \qquad \dfrac{dy}{dx} = \dfrac{1}{1 - x^2}$

41. $y = \sin^{-1}\dfrac{\sqrt{1 + x} - \sqrt{1 - x}}{\sqrt{1 + x} + \sqrt{1 - x}}$

$$\frac{dy}{dx} = \frac{\sqrt{1 - \sqrt{1 - x^2}}}{x\sqrt{2}\,(1 - x^2)^{\frac{1}{4}}}$$

42. $y = \dfrac{1}{x\log x} \qquad \dfrac{dy}{dx} = -\dfrac{1 + \log x}{(x\log x)^2}$

43. $y = \cos^{-1}\left(\frac{a+x}{2a+x}\right)$

$$\frac{dy}{dx} = -\frac{a}{\sqrt{(3a^2+2ax)}(2a+x)}$$

44. $y = \frac{\sqrt{ax(bx+c)}}{\sqrt{x+a}}$

$$\frac{dy}{dx} = \sqrt{\frac{ax(bx+c)}{(x+a)}}\,\frac{1}{2}\left\{\frac{1}{x}+\frac{b}{bx+c}-\frac{1}{x+a}\right\}$$

45. $y = \{x+\sqrt{(x^2-1)}\}^n$

$$\frac{dy}{dx} = n(x+\sqrt{x^2-1})^{n-1}\left\{1+\frac{x}{\sqrt{x^2-1}}\right\}$$

46. $y = \tan^{-1}\left(\frac{x+a^2}{a-ax}\right)+\log(x^2+a^2)$ $\quad \frac{dy}{dx} = \frac{a+2x}{x^2+a^2}$

47. $y = (a^{\frac{1}{3}}+x^{\frac{1}{3}})^n(b^3+x^3)^m$

$$\frac{dy}{dx} = (a^{\frac{1}{3}}+x^{\frac{1}{3}})^{n-1}(b^3+x^3)^{m-1}\left\{3mx^2(a^{\frac{1}{3}}+x^{\frac{1}{3}})+\frac{nx^{-\frac{2}{3}}}{3}(b^3+x^3)\right\}$$

48. $y = e^{\frac{1}{x}}$ $\quad \frac{dy}{dx} = -\frac{e^{\frac{1}{x}}}{x^2}$

49. $y = x^{\frac{x}{a}}$ $\quad \frac{dy}{dx} = \frac{x^{\frac{x}{a}}}{a}(1+\log x)$

50. $y = \cos(\log\tan^{-1}x)$ $\quad \frac{dy}{dx} = -\frac{\sin(\log\tan^{-1}x)}{\tan^{-1}x(1+x^2)}$

51. $y = \frac{x}{\sqrt{(1+x^2)}}$ $\quad \frac{dy}{dx} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$

52. $y = \log\tan\frac{x}{2}-\frac{\cos x}{\sin^2 x}$ $\quad \frac{dy}{dx} = \frac{2}{\sin^3 x}$

53. $y = \tan\frac{x}{2}\sin 2(x-a)$

$$\frac{dy}{dx} = \frac{\sin x\cos 2(x-a)+\sin 2(x-a)}{1+\cos x}$$

54. $y = xe^{3x} \qquad \frac{dy}{dx} = e^{3x}(1+3x)$

55. $y = \frac{x^{\frac{1}{3}} \sin \frac{x}{3}}{1+x} \quad \frac{dy}{dx} = \frac{x^{\frac{1}{3}} \cos \frac{x}{3} + x^{-\frac{2}{3}} \sin \frac{x}{3}}{3(1+x)} - \frac{x^{\frac{1}{3}} \sin \frac{x}{3}}{(1+x)^2}$

56. $y = \left(\frac{x^x}{a^a}\right)^{\frac{1}{x-a}}$

$$\frac{dy}{dx} = \left(\frac{x^x}{a^a}\right)^{\frac{1}{x-a}} \left(\frac{(x-a)(1+\log x) - \log\left(\frac{x^x}{a^a}\right)}{(x-a)^2}\right)$$

57. $y = x^x(1-x)^{1-x} \quad \frac{dy}{dx} = x^x(1-x)^{1-x} \left(\log\left(\frac{x}{1-x}\right)\right)$

58. $y = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \quad \frac{dy}{dx} = \sqrt{(a^2-x^2)}$

59. $y = (\sin x \sin 2x \sin 3x)^n$

$$\frac{dy}{dx} = ny\,(\cot x + 2 \cot 2x + 3 \cot 3x)$$

60. $y = \tan^{-1}(a^x) \qquad \frac{dy}{dx} = \frac{a^x \log a}{1+a^{2x}}$

61. $y = \sqrt{(ax+x^2)} + a \log(\sqrt{x} + \sqrt{a+x})$

$$\frac{dy}{dx} = \sqrt{\left(\frac{x+a}{x}\right)}$$

62. $y = \log \sqrt{\left(\frac{a \sin x + b \cos x}{a \sin x - b \cos x}\right)}$

$$\frac{dy}{dx} = -\frac{ab}{a^2 \sin^2 x - b^2 \cos^2 x}$$

63. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$$\frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f}$$

64. $y = \cfrac{\theta^2}{1 + \cfrac{\theta^2}{1 + \text{etc.}}} \qquad \frac{dy}{d\theta} = \frac{\theta}{\sqrt{\theta^2 + \frac{1}{4}}}$

65. $y = e^{\frac{y-x}{x}}$ $\qquad \frac{dy}{dx} = \frac{(1 + \log y)^2}{\log y}$

66. Given $y = a^4b^3x^2$ find the rate of change of $y$ (1) with regard to $a$ (2) with regard to $b$ (3) with regard to $x$

$$\frac{dy}{da} = 4a^3b^3x^2 \qquad \frac{dy}{db} = 3a^4b^2x^2 \qquad \frac{dy}{dx} = 2a^4b^3x$$

67. $y = x\sqrt{x\sqrt{x\sqrt{x}}} \ldots \qquad \frac{dy}{dx} = 2x$

## CHAPTER IV

# SUCCESSIVE DIFFERENTIATION

### Successive Derived Functions and Differentials

In the preceding chapter we have shown how to find the rate of change of a function of one variable with regard to the variable. That is we have shown how to find what is called the *first derived function* from the original function.

Suppose $y$ to be equal to any expression involving $x$, and denoting this expression by $f(x)$ we have

$$y = f(x)$$

When $x$ is increased by an indefinitely small increment $dx$, let $y$ become $y + dy$

$$\therefore y + dy = f(x + dx)$$
$$\therefore \frac{dy}{dx} = \frac{f(x + dx - f(x)}{dx}$$

the limiting value of this expression when $dx$ is indefinitely small is called the differential coefficient, or first derived function, of $y$ or $f(x)$ with regard to $x$ and is generally denoted by $f'(x)$.

$$\therefore \frac{dy}{dx} = f'(x) \text{ the first derived function,}$$

If $f'(x)$ contains $x$ we can differentiate it again thus

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{df'(x)}{dx} = f''(x) \text{ the second derived function}$$

Instead of $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ we may express it in the form $\frac{d^2y}{dx^2}$ therefore if $y = f(x)$ be differentiated once we have

$$\frac{dy}{dx} = f'(x)$$

On differentiating again we have

$$\frac{d^2y}{dx^2} = f''(x)$$

On differentiating a third time we have

$$\frac{d^3y}{dx^3} = f'''(x)$$

and on differentiating $f(x)$ $n$ times we have

$$\frac{d^ny}{dx^n} = f^n(x)$$

$f^n(x)$ is called the $n$th differential coefficient of $f(x)$ with regard to $x$

An example will illustrate this.

If
$$y = x^3 - 4x^2 + 5x - 7 = f(x)$$
$$\frac{dy}{dx} = 3x^2 - 8x + 5 = f'(x)$$
$$\frac{d^2y}{dx^2} = 6x - 8 = f''(x)$$
$$\frac{d^3y}{dx^3} = 6 = f'''(x)$$
and
$$\frac{d^4y}{dx^4} = 0 = f''''(x)$$

In this example the fourth derived function is zero, since the third derived function is a constant. Similarly it may be

shown that the $(n+1)$th derived function of $ax^n + bx^{n-1} +$ etc., is zero, where $n$ is a positive integer.

If $n$ be negative or fractional there is no derived function in which some power of $x$ will not appear and consequently no derived function will vanish.

$$
\begin{array}{llll}
\text{If} & y & = f(x) & \\
 & dy & = f'(x)dx & \text{the first differential} \\
\text{and} & d^2y & = f''(x)dx^2 & \text{,, second ,,} \\
\cdot & \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\
 & d^ny & = f^n(x)dx^n & \text{,, } n\text{th ,,}
\end{array}
$$

If $s = f(t)$ where $s$ denotes displacement and $t$ time, then $\frac{ds}{dt} = f'(t) =$ velocity, that is rate of change of space with regard to time, and $\frac{d^2s}{dt^2} = f''(t) =$ acceleration or rate of change of velocity.

If $y = e^{ax}$ then $\frac{dy}{dx} = ae^{ax}$

$$\frac{d^2y}{dx^2} = a^2e^{ax} \text{ then } \frac{d^3y}{dx^3} = a^3e^{ax}$$

and $$\frac{d^ny}{dx^n} = a^ne^{ax}$$

If $$y = e^x \text{ then } \frac{d^ny}{dx^n} = e^x$$

If $$y = \sin pt$$

$$\frac{dy}{dt} = p \cos pt = p \sin\left(pt + \frac{\pi}{2}\right)$$

$$\frac{d^2y}{dt^2} = p^2 \cos\left(pt + \frac{\pi}{2}\right) = p^2 \sin(pt + \pi)$$

and $$\frac{d^ny}{dt^n} = p^n \sin\left(pt + \frac{n\pi}{2}\right)$$

Similarly if $y = \cos pt$

$$\frac{d^ny}{dt^n} = p^n \cos\left(pt + \frac{n\pi}{2}\right)$$

Given $y = e^{ax} \sin bx$

$$\frac{dy}{dx} = ae^{ax} \sin bx + be^{ax} \cos bx$$
$$= e^{ax}(a \sin bx + b \cos bx)$$
$$= e^{ax}\sqrt{a^2 + b^2}\{\sin bx \cos \theta + \cos bx \sin \theta\}$$
$$= e^{ax}(a^2 + b^2)^{\frac{1}{2}} \sin (bx + \theta)$$

where

$$\theta = \tan^{-1} \frac{b}{a}$$

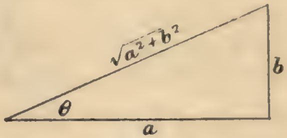


FIG. 23.

On differentiating a second time we have

$$\frac{d^2y}{dx^2} = e^{ax}(a^2 + b^2)^{\frac{2}{2}}\{\sin\ bx + 2\theta\}$$

Similarly on differentiating $n$ times we obtain

$$\frac{d^ny}{dx^n} = e^{ax}(a^2 + b^2)^{\frac{n}{2}} \sin (bx + n\theta)$$

In an electric circuit let E be the impressed E.M.F., R the resistance and L the inductance, and I the current,

then $$E = RI + L\frac{dI}{dt} \quad . \quad . \quad . \quad . \quad (1)$$

Suppose I to be the instantaneous value of the current and $I_0$ the maximum value, we have $I = I_0 \sin pt$ where $p = 2\pi \times$ frequency,

$$\frac{dI}{dt} = I_0 p \cos pt$$

Substituting in (1) we have

$$E = RI_0 \sin pt + LI_0p \cos pt$$
$$= I_0\{R \sin pt + Lp \cos pt\}$$
$$= I_0\sqrt{R^2 + (Lp)^2} \sin (pt + \theta) \text{ where } \tan \theta = \frac{Lp}{R}$$

$\theta$ being the angle by which the E.M.F. must lead the current if the current obeys the law $I = I_0 \sin pt$

In order to represent this graphically, plot a sine curve (*see* the author's *Elementary Practical Mathematics*, page 163) whose amplitude is $RI_0$ to represent the electromotive-force required to drive the current through the resistance if there were no inductance in the circuit. Then on the same time base plot a cosine curve whose amplitude is $LI_0p$ to

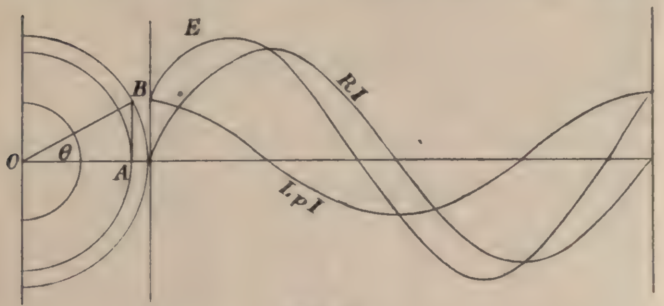


FIG. 24.

represent the electromotive-force necessary to balance the reaction of self-induction. Also plot a curve whose amplitude is $\sqrt{(RI_0)^2 + (LI_0)^2}$ as shown in the accompanying figure.

The E curve represents the electromotive-force that must be impressed on the circuit in order to produce the current I

In the figure $OA = RI_0$ $AB = LpI_0$ $OB = \sqrt{(RI_0)^2 + (LpI_0)^2}$, and the angle AOB is the angle of lag or amount by which the current lags behind the electromotive-force.

The expression $\sqrt{R^2 + (Lp)^2}$ is called the *impedance.*

*Example.* Given $R = 2$, $L = \cdot 002$, $I = 50$ virtual amperes, and $T = \dfrac{1}{50}$, find E, the virtual volts that

must be impressed on the circuit, $p$, angle of lag $\theta$, and the impedance.

Here the virtual volts, $E = 50\sqrt{2^2 + (\cdot 628)^2} = 104{\cdot}8$

and $\qquad p = \dfrac{2\pi}{T} = 314{\cdot}16$

Also $\qquad \tan\theta = \dfrac{Lp}{R} = \dfrac{\cdot 628}{2} = \cdot 314$

therefore $\qquad \theta = 17^\circ\ 26'$

The impedance $= \sqrt{2^2 + (\cdot 628)^2} = 2{\cdot}096$

In the figure A represents the position of the cross-head of a steam engine when the crank is at the dead centre. B

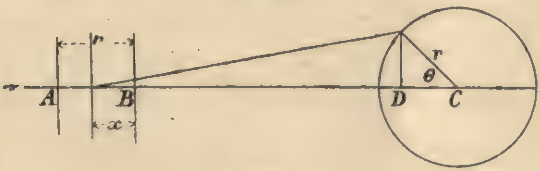


Fig. 25.

is the middle point of the stroke of the cross-head and $x$ is the displacement of the cross-head from its mean position when the crank angle is $\theta$

$$\text{Now AC} = l + r = r - x + \sqrt{l^2 - r^2\sin^2\theta} + r\cos\theta$$

$$\therefore x = r\cos\theta + \sqrt{l^2 - r^2\sin^2\theta} - l \quad . \quad . \quad (1)$$

where $l$ is the length of the connecting rod, $r$ is the crank radius and $\theta$ is the crank angle measured from the dead centre.

The piston velocity for any given crank angle is obtained by differentiating $x$ with regard to time $t$ thus—

$$v = \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt} - \frac{r^2\sin\theta\cos\theta}{\sqrt{l^2 - r^2\sin^2\theta}}\frac{d\theta}{dt}$$

The piston acceleration is the rate of change of velocity.

$$\text{Acceleration} \quad = \quad \frac{dv}{dt} \quad = \quad \frac{d^2x}{dt^2}$$

$$= \quad -r\left[\cos\theta + r\left\{\frac{l^2\cos 2\theta + r^2\sin^4\theta}{(l^2 - r^2\sin^2\theta)^{\frac{3}{2}}}\right\}\right]\left(\frac{d\theta}{dt}\right)^2$$

This is assuming the angular velocity of the crank to be constant.

The following investigation is an approximate method of finding the piston velocity and acceleration.

$$x \quad = \quad r\cos\theta + \sqrt{l^2 - r^2\sin^2\theta} - l$$

$$= \quad r\cos\theta + l\left\{1 - \frac{r^2}{l^2}\sin^2\theta\right\}^{\frac{1}{2}} - l$$

$$= \quad r\cos\theta + l\left\{1 - \frac{r^2}{2l^2}\sin^2\theta - \frac{1}{8}\,\frac{r^4}{l^4}\sin^4\theta - \text{etc.}\right\} - l$$

In practice $\frac{r}{l}$ is a fraction usually less than $\frac{1}{4}$ and therefore $\frac{r^4}{l^4}$ is a small fraction which has to be multiplied by $\sin^4\theta$, therefore all the terms beyond the second inside the double bracket may be neglected, and therefore

$$x \quad = \quad r\cos\theta + l - \frac{r^2}{2l}\sin^2\theta - l$$

$$\therefore x \quad = \quad r\cos\theta - \frac{r^2}{4l}(1 - \cos 2\theta). \quad . \quad . \quad (2)$$

This result shows that the motion of the piston of an engine is approximately the motion compounded of two cranks, one going at the same speed as the engine and another going at double the speed, the amplitude of the first being $r$ and that of the second being $\frac{r^2}{4l}$

On differentiating $x$ with regard to $t$ we get the piston velocity

$$v = \frac{dx}{dt} = -r\left\{\sin\theta + \frac{r}{2l}\sin 2\theta\right\}\frac{d\theta}{dt}$$

On differentiating again we get the piston acceleration

$$a = \frac{d^2x}{dt^2} = -r\left\{\cos\theta + \frac{r}{l}\cos 2\theta\right\}\left(\frac{d\theta}{dt}\right)^2$$

*Example.*—Suppose that $r = 1$ and $l = 4$ and let the speed of the engine be 120 revolutions per minute, it is required to find the velocity and the acceleration of the piston when the crank angle is 60° Here $\frac{d\theta}{dt} = 4\pi$

$$v = -\left\{\sin 60^\circ + \frac{1}{8}\sin 120^\circ\right\}4\,\pi = -\frac{9\sqrt{3}}{4}\pi$$

and
$$a = -\left\{\cos 60^\circ + \frac{r}{l}\cos 120^\circ\right\}(4\pi)^2$$
$$= -\left\{\frac{1}{2} - \frac{1}{4}\times\frac{1}{2}\right\}(4\pi)^2 = -\frac{3}{8}\times 16\pi^2 = -6\pi^2$$

Observe that the piston velocity varies as the angular velocity but the piston acceleration varies as the square of the angular velocity.

## Examples on Successive Differentiation

√ 1. If $y = x^4$ show that $\frac{d^3y}{dx^3} = 24x$

√ 2. If $y = x^{\frac{1}{2}}$ ,, $\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}}$

√ 3. If $y = x^{-\frac{3}{2}}$ ,, $\frac{d^4y}{dx^4} = \frac{945}{16}x^{-\frac{11}{2}}$

4. If $y = \sin x$ show that $\dfrac{d^6y}{dx^6} = -\sin x$

5. If $y = \cos x^2$ „ $\dfrac{d^3y}{dx^3} = 8x^3 \sin x^2 - 12x \cos x^2$

6. If $y = x \sin x$ „ $\dfrac{d^5y}{dx^5} = 5 \sin x + x \cos x$

7. If $y = x^n e^x$

show that $$\frac{d^2y}{dx^2} = e^x\{n(n-1)x^{n-2} + 2nx^{n-1} + x^n\}$$

8. If $y = e^x \tan^{-1} x$

show that $$\frac{d^2y}{dx^2} = \frac{(1-x)^2e^x}{(1+x^2)^2} + \frac{e^x}{1+x^2} + y$$

9. If $y = e^{x^2}$

show that $$\frac{d^6y}{dx^6} = e^{x^2}(64x^6 + 480x^4 + 720x^2 + 120)$$

10. If $y = \log(\cos x)$

show that $$\frac{d^3y}{dx^3} = -2 \sec^2 x \tan x$$

11. If $y = e^x \sin x$

show that $$\frac{d^4y}{dx^4} + 4y = 0$$

12. If $y = e^{ax} \sin x$

show that $$\frac{d^ny}{dx^n} = \frac{e^{ax} \sin(x + n\theta)}{\sin^n \theta}$$

where $\tan \theta = \dfrac{1}{a}$

13. If $y = (x^2 + n^2) \tan^{-1}\left(\dfrac{x}{n}\right)$

show that $$\frac{d^3y}{dx^3} = \frac{4n^3}{(n^2 + x^2)^2}$$

14. If $y = C \sin kx + C_1 \cos kx$

show that $\dfrac{d^4y}{dx^4} = k^4y$

15. If $y = e^{k \sin^{-1} x}$

show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = k^2y$

16. If $y = k \cos (\log x) + l \sin (\log x)$

show that $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

17. $y = \dfrac{n^2}{n^2 + x^2}$

$$\frac{d^5y}{dx^5} = \frac{-48n^2x(9n^4 - 26n^2x^2 + 5x^4)}{(n^2 + x^2)^6}$$

18. Find the velocity and acceleration of the piston of a steam engine from the following data :—

Crank, 12 inches; connecting rod, 5 feet; crank angle, 30°; speed, 100 revolutions per minute.

The angular velocity $\dfrac{d\theta}{dt} = \dfrac{10\pi}{3}$ radians per second.

Velocity = 368·6 feet per minute.
Acceleration = 105·9 feet per second per second.

19. Given $x = r \cos \theta - \dfrac{r^2}{4l}(1 - \cos 2\theta)$ find the piston velocity and acceleration where $r = 1$ and $l = 4$, $\dfrac{d\theta}{dt} = 2\pi$

(1) When $\theta = 0$ *Ans.* (1) $v = 0$
Acc. $= 5\pi^2$ ft. p.s.p.s.

(2) „ $\theta = \pi$ *Ans.* (2) $v = 0$
Acc. $= 3\pi^2$ ft. p.s.p.s.

(3) „ $\theta = 60°$ *Ans.* (3) $v = \dfrac{9\sqrt{3}\pi}{8}$
Acc. $= \dfrac{3\pi^2}{2}$ ft. p.s.p.s.

## CHAPTER V

# EXPANSION OF FUNCTIONS

### Taylor's and Maclaurin's Theorems

If $f(x+y)$ be a continuous function of $x+y$ which admits of being expressed in the form of a series of ascending integral powers of $x$ or $y$ then

$$f(x+y) = A + By + Cy^2 + Dy^3 + \text{etc.}$$

or $$f(x+y) = A_{,} + B_{,}x + C_{,}x^2 + D_{,}x^3 + \text{etc.}$$

where A B C D etc., are functions of $x$ and independent of $y$ and $A_{,}$ $B_{,}$ $C_{,}$ $D_{,}$ etc., are functions of $y$ and independent of $x$ For example:

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{\underline{|2}}x^{n-2}y^2 + \text{etc.}$$
$$= A + By + Cy^2 + \text{etc.}$$

where

$$A = x^n, \ B = nx^{n-1} \text{ and } C = \frac{n(n-1)}{\underline{|2}}x^{n-2} \text{ etc.}$$

We shall now show how the values of A B C D etc., or $A_{,}$ $B_{,}$ $C_{,}$ $D_{,}$ etc., may be obtained in terms of $x$ or $y$ respectively.

If $f(x+y)$ be a function which admits of being expanded in powers of $x$ or $y$; on differentiating it with respect to $x$ (treating $y$ as constant) we get a result which is the same as if we were to differentiate it with regard to $y$ treating $x$ as constant.

For the proof of this statement,

Let $u = f(x+y) = f(z)$

where $z = x + y$

Then $\dfrac{du}{dz} = f'(z)$

but $\dfrac{du}{dx} = \dfrac{du}{dz}\dfrac{dz}{dx}$

and since $z = x + y$

we have $\dfrac{dz}{dx} = 1$ treating $y$ as a constant

therefore $\dfrac{du}{dx} = f'(z)$

Similarly, $\dfrac{du}{dy} = \dfrac{du}{dz}\dfrac{dz}{dy} = f'(z)$

Since $\dfrac{dz}{dy} = 1$ treating $x$ as a constant,

therefore $\dfrac{du}{dx} = \dfrac{du}{dy}$, where $u = f(x+y)$

For example: Let

$$u = (x+y)^4$$

therefore $\dfrac{du}{dx} = 4(x+y)^3$ treating $y$ as constant

and $\dfrac{du}{dy} = 4(x+y)^3$ treating $x$ as constant;

therefore $\dfrac{du}{dx} = \dfrac{du}{dy}$

Let

$$u = f(x+y) = A + By + Cy^2 + Dy^3 + \text{etc.} \quad (1)$$

On differentiating this with regard to $x$, we get

$$\frac{du}{dx} = \frac{dA}{dx} + \frac{dB}{dx}y + \frac{dC}{dx}y^2 + \frac{dD}{dx}y^3 + \text{etc.} \quad (2)$$

also $\dfrac{du}{dy} = B + 2Cy + 3Dy^2 + \text{etc.}$

and since we have proved that

$$\frac{du}{dx} = \frac{du}{dy}$$

we are at liberty to assume that the coefficients of like powers of $y$ are equal in the above equations, therefore

$$\frac{dA}{dx} = B, \quad \frac{dB}{dx} = 2C \quad \text{and} \quad \frac{dC}{dx} = 3D \text{ etc.}$$

Again, since (1) holds for all values of $y$, it must hold when $y = 0$, therefore

$$f(x) = A$$

Again, $$B = \frac{dA}{dx} = f'(x)$$

and $$C = \frac{1}{2}\frac{dB}{dx} = \frac{1}{\underline{|2}} f''(x)$$

Similarly, $$D = \frac{1}{\underline{|3}} f'''(x) \text{ etc.}$$

On substituting these values of A, B, C, D, etc., in (1) we have

$$f(x+y) = f(x) + yf'(x) + \frac{y^2}{\underline{|2}} f''(x) + \frac{y^3}{\underline{|3}} f'''(x) + \text{etc.}$$

This result is called **Taylor's Theorem.**

Similarly, we may show that

$$f(x+y) = f(y) + xf'(y) + \frac{x^2}{\underline{|2}} f''(y) + \frac{x^3}{\underline{|3}} f'''(y) + \text{etc.}$$

We shall now apply *Taylor's Theorem* in expanding a few functions of $x + y$

**Expansion of** $(x+y)^6$

$$(x+y)^6 = f(x) + yf'(x) + \frac{y^2}{\underline{|2}} f''(x) + \frac{y^3}{\underline{|3}} f'''(x) + \text{etc.}$$

Here

$$f(x) = x^6,\ f'(x) = 6x^5 \text{ and } f''(x) = 6 \times 5x^4, \text{ etc.}$$

On substituting these values for the several successive derived functions of $x$ and simplifying the coefficients, we have

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

**To expand** $\log(x+y)$

$$\text{Log}(x+y) = f(x) + yf'(x) + \frac{y^2}{\underline{|2}}f''(x) + \frac{y^3}{\underline{|3}}f'''(x) + \text{etc.}$$

Here

$$f(x) = \log x,\ f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\ f'''(x) = \frac{2}{x^3} \text{ etc.}$$

On substituting these values for the several derived functions, we get

$$\text{Log}(x+y) = \log x + \frac{y}{x} - \frac{y^2}{2x^2} + \frac{y^3}{3x^3} - \frac{y^4}{4x^4} + \text{etc.}$$

**To expand** $\sin(x+y)$

$$\text{Sin}(x+y) = f(x+y) = f(x) + yf'(x) + \frac{y^2}{\underline{|2}}f''(x) + \frac{y^3}{\underline{|3}}f'''(x) + \text{etc.}$$

Here

$$f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x \text{ etc.}$$

On substituting these values for the several derived functions of $x$ we obtain

$$\sin(x+y) = \sin x + y\cos x - \frac{y^2}{\underline{|2}}\sin x - \frac{y^3}{\underline{|3}}\cos x + \frac{y^4}{\underline{|4}}\sin x + \frac{y^5}{\underline{|5}}\cos x - \text{etc.}$$

or thus

$$\operatorname{Sin}(x+y) = \sin x\left(1-\frac{y^2}{\underline{|2}}+\frac{y^4}{\underline{|4}}-\frac{y^6}{\underline{|6}}+\text{etc.}\right)$$
$$+\cos x\left(y-\frac{y^3}{\underline{|3}}+\frac{y^5}{\underline{|5}}-\frac{y^7}{\underline{|7}}+\text{etc.}\right)$$

**To expand** $\cos(x+y)$

$$\operatorname{Cos}(x+y) = f(x+y) = f(x)+yf'(x)+\frac{y^2}{\underline{|2}}f''(x)+\text{ etc.}$$

Here

$$f(x = \cos x,\ f'(x) = -\sin x,\ f''(x) = -\cos x \quad \text{etc.}$$

On substituting, we obtain

$$\operatorname{Cos}(x+y) = \cos x - y\sin x - \frac{y^2}{\underline{|2}}\cos x + \frac{y^3}{\underline{|3}}\sin x + \text{ etc.}$$
$$= \cos x\left(1-\frac{y^2}{\underline{|2}}+\frac{y^4}{\underline{|4}}-\frac{y^6}{\underline{|6}}+\text{etc.}\right)$$
$$-\sin x\left(y-\frac{y^3}{\underline{|3}}+\frac{y^5}{\underline{|5}}-\frac{y^7}{\underline{|7}}+\text{etc.}\right)$$

**To expand** $a^{x+y}$

$$a^{x+y} = f(x+y) = f(x)+yf'(x)+\frac{y^2}{\underline{|2}}f''(x)+\text{ etc.}$$

Here

$$f(x) = a^x,\ f'(x) = a^x\log a \text{ and } f''(x) = a^x(\log a)^2 \quad \text{etc.}$$

Therefore

$$a^{x+y} = a^x + ya^x\log a + \frac{(y\log a)^2}{\underline{|2}}a^x + \text{etc.}$$
$$= a^x\left(1+y\log a+\frac{(y\log a)^2}{\underline{|2}}+\frac{(y\log a)^3}{\underline{|3}}+\text{etc.}\right)$$

## Maclaurin's Theorem.

In the expansion of $f(x+y)$ by *Taylor's Theorem*, if we put $x = 0$, we have then

$$f(y) = f(0+y) = f(0) + yf'(0) + \frac{y^2}{\underline{|2}}f''(0)$$
$$+ \frac{y^3}{\underline{|3}}f'''(0) + \text{etc.}$$

Similarly, we may show that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}}f''(0) + \frac{x^3}{\underline{|3}}f'''(0) + \text{etc.}$$

This result is called **Maclaurin's Theorem.**

The beginner has usually some difficulty in interpreting the meaning of $f(0)$, $f'(0)$ and $f''(0)$ etc.

Now

| | | | | |
|---|---|---|---|---|
| $f(0)$ is what | $f(x)$ | becomes when $x$ | $=$ | $0$ |
| $f'(0)$ ,, | $f'(x)$ | ,, | | ,, |
| $f''(0)$ ,, | $f''(x)$ | ,, | | ,, |

Expansion of sin $x$

Here

$$f(x) = \sin x \qquad \therefore f(0) = \sin 0 = 0$$

and

$$f'(x) = \cos x \qquad \therefore f'(0) = \cos 0 = 1$$
$$f''(x) = -\sin x \qquad \therefore f''(0) = -\sin 0 = 0$$
$$f'''(x) = -\cos x \qquad \therefore f'''(0) = -\cos 0 = -1$$

$$\therefore \text{Sin } x = f(x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}}f''(0)$$
$$+ \frac{x^3}{\underline{|3}}f'''(0) + \text{etc.}$$
$$= 0 + x \times 1 + \frac{x^2}{\underline{|2}} \times 0 + \frac{x^3}{\underline{|3}} \times (-1) + \frac{x^4}{\underline{|4}} \times 0 + \text{etc.}$$

$$\therefore \sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \text{etc.}$$

$$\text{Cos } x = f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \text{etc.}$$

Here

$$f(x) = \cos x \qquad \therefore f(0) = \cos 0 = 1$$

and

$$f'(x) = -\sin x \qquad \therefore f'(0) = -\sin 0 = 0$$
$$f''(x) = -\cos x \qquad \therefore f''(0) = -\cos 0 = -1$$

$$\therefore \cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \text{etc.}$$

**Exponential Theorem.**

**To expand** $a^x$

Here

$$a^x = f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \text{etc.}$$

Now

$$f(0) = a^0 = 1 \text{ and } f'(0) = a^0 \log a = \log a$$

while

$$f''(0) = a^0 (\log a)^2 = (\log a)^2 \quad \text{etc.}$$

$$\therefore a^x = 1 + x \log a + \frac{x^2}{\lfloor 2}(\log a)^2 + \frac{x^3}{\lfloor 3}(\log a)^3 + \text{etc.}$$

If we substitute $e$ (the base of the Napierian logarithms) for $a$ we get

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{etc., since } \log_e e = 1$$

**To expand** $\tan x$

$$f(x) = \tan x = f(0) + xf'(0) + \frac{x^2}{\lfloor 2}f''(0) + \frac{x^3}{\lfloor 3}f'''(0) + \frac{x^4}{\lfloor 4}f''''(0) + \text{etc.}$$

$$
\begin{aligned}
f(x) &= \tan x \\
&\quad \therefore f(0) = \tan(0) = 0 \\
f'(x) &= 1 + \tan^2 x \\
&\quad \therefore f'(0) = 1 + \tan^2(0) = 1 \\
f''(x) &= 2\tan x + 2\tan^3 x \\
&\quad \therefore f''(0) = 2\tan(0) + 2\tan^3(0) = 0 \\
f'''(x) &= 2 + 8\tan^2 x + 6\tan^4 x \\
&\quad \therefore f'''(0) = 2 \\
f''''(x) &= 16\tan x + 40\tan^3 x + 24\tan^5 x \\
&\quad \therefore f''''(0) = 0 \\
f'''''(x) &= 16 + \text{terms involving } \tan x \\
&\quad \therefore f'''''(0) = 16 \text{ etc.}
\end{aligned}
$$

Therefore

$$\tan x = 0 + x + \frac{x^2}{\underline{|2}} \times 0 + \frac{x^3}{\underline{|3}} \times 2 + \frac{x^4}{\underline{|4}} \times 0 + \frac{x^5}{\underline{|5}} \times 16 + \text{etc.};$$

that is $$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \text{etc.}$$

**Expansion of log $(1 + x)$**

$$f(x) = \log(1 + x) = f(0) + xf'(0) + \frac{x^2}{\underline{|2}} f''(0) + \frac{x^3}{\underline{|3}} f'''(0) + \text{etc.}$$

Here
$$
\begin{aligned}
f(0) &= \log 1 = 0 \\
f'(0) &= \frac{1}{1+0} = 1 \\
f''(0) &= -\frac{1}{(1+0)^2} = -1 \\
f'''(0) &= \frac{2}{(1+0)^3} = 2
\end{aligned}
$$

$$\therefore \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \text{etc.} \quad . \quad . \quad (1)$$

In this result let $x$ be changed to $-x$

$$\therefore \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \text{etc.} \quad . \quad (2)$$

Subtract (2) from (1) and we have

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \frac{x^9}{9} + \text{etc.}\right) \quad . \quad (3)$$

$$\text{Assume} \quad \frac{1+x}{1-x} = \frac{n+1}{n} \qquad \therefore x = \frac{1}{2n+1}$$

and (3) transforms to

$$\log_e\left(\frac{1+n}{n}\right) = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \frac{1}{7(2n+1)^7} + \text{etc.}\right\}$$

$$\therefore \log_e(1+n) = \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \text{etc.}\right\} \quad . \quad . \quad . \quad (4)$$

This is a series by means of which logarithms to the base $e$ may be calculated, that is, *Hyperbolic* or *Napierian logarithms.*

Let $n = 1$ in (4) and we have

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3} \times \frac{1}{(3)^3} + \frac{1}{5} \times \frac{1}{3^5} + \frac{1}{7} \times \frac{1}{(3)^7} \text{ etc.}\right\}$$

Again, let $n = 2$ and we get

$$\log_e 3 = \log_e 2 + 2\{\tfrac{1}{5} + \tfrac{1}{3}(\tfrac{1}{5})^3 + \tfrac{1}{5}(\tfrac{1}{5})^5 + \tfrac{1}{7}(\tfrac{1}{5})^7 + \ldots\}$$

and $\log_e 4 = 2 \log_e 2.$

To get $\log_e 5$ let $n = 4$, $\log_e 6 = \log_e 3 + \log_e 2$

This shows how logarithms to the base $e$ may be calculated.

The common logarithms may be obtained from the Hyperbolic logarithms by multiplying by $\dfrac{1}{\log_e 10}$ which is $\cdot 43429 \ldots$

**Expansion of $\tan^{-1} x$**

Assume $\tan^{-1} x = a + bx + cx^2 + dx^3 +$ etc., and differentiate both sides of this equation with regard to $x$ therefore

$$\frac{1}{1+x^2} = b + 2cx + 3dx^2 + \text{etc.} \quad . \quad . \quad (\alpha)$$

Now
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \text{etc.} \quad . \quad (\beta)$$

and since $(\alpha)$ and $(\beta)$ are identical, we may assume that the coefficients of like powers of $x$ are equal in both expressions; therefore $b = 1$, $c = 0$, $d = -\frac{1}{3}$, and $a = 0$, since $\tan^{-1}(0) = 0$

$$\therefore \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{etc.} \quad . \quad (\gamma)$$

By means of $(\gamma)$ we may calculate the numerical value of $\pi$.

Let $x = 1$ and we have

$$\tan^{-1} 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \text{etc.}$$

$$= 1 - 2\left(\frac{1}{15} + \frac{1}{63} + \frac{1}{143} + \text{etc.}\right)$$

$$\therefore \pi = 4\left\{1 - 2\left(\frac{1}{15} + \frac{1}{63} + \frac{1}{143} + \ldots\right)\right\}$$

$$= 3{\cdot}1416 \ldots$$

**Euler's Formulæ for Sine and Cosine.**

If we expand $e^{x\sqrt{-1}}$ by Maclaurin's Theorem we obtain

$$e^{x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \text{etc.} \ldots$$
$$+ \sqrt{-1}\left\{x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \text{etc.}\right\}$$
$$= \cos x + \sqrt{-1} \sin x$$

Change $x$ to $-x$ and we have

$$e^{-x\sqrt{-1}} = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \text{etc.}$$
$$- \sqrt{-1}\left\{x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \text{etc.}\right\}$$
$$= \cos x - \sqrt{-1} \sin x$$

On adding and dividing by 2 we get

$$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}$$

Subtracting, we get

$$\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}}$$

**To expand** $\sin^{-1} x$

Assume

$$\sin^{-1} x = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.} \quad . \quad (a)$$

Differentiate both sides; thus

$$\frac{1}{(1-x^2)^{\frac{1}{2}}} = b + 2cx + 3dx^2 + 4ex^3 + \text{etc.} \quad . \quad (\beta)$$

But

$$\frac{1}{(1-x^2)^{\frac{1}{2}}} = 1 + \frac{1}{2}x^2 + \frac{1 \times 3}{2 \times 4}x^4 + \frac{1 \times 3 \times 5}{2 \times 4 \times 6}x^6$$
$$+ \frac{1 \times 3 \times 5 \times 7}{2 \times 4 \times 6 \times 8}x^8 + \text{etc.} \quad . \quad . \quad (\gamma)$$

Equating the coefficients of like powers of $x$ in ($\beta$) and ($\gamma$) we get $b = 1$, $c = 0$, $d = \frac{1}{2 \times 3}$, $e = 0$, etc., and $a = 0$, $\sin^{-1}(0) = 0$

On substituting these values for $a$ $b$ $c$ $d$ etc., in ($\alpha$) we have

$$\sin^{-1} x = x + \frac{1}{2 \times 3} x^3 + \frac{1 \times 3}{2 \times 4 \times 5} x^5 + \frac{1 \times 3 \times 5}{2 \times 4 \times 6 \times 7} x^7 + \text{etc.}$$

In the above trigonometrical expansions the angle $x$ is estimated in circular measure, and in order that the expansions may form rapidly convergent series, the value of $x$ must be less than unity.

*Examples.*

**To find the sine and cosine of 10°**

We must express 10° in radians, that is, $\frac{\pi}{18}$ radians; therefore

$$\sin 10^\circ = \frac{\pi}{18} - \frac{1}{\underline{|3}} \left(\frac{\pi}{18}\right)^3 + \frac{1}{\underline{|5}} \left(\frac{\pi}{18}\right)^5 - \text{etc.}$$

*Ans.* $\sin 10^\circ = \cdot 1736482 \ldots$

$$\cos 10^\circ = 1 - \frac{1}{2} \left(\frac{\pi}{18}\right)^2 + \frac{1}{\underline{|4}} \left(\frac{\pi}{18}\right)^4 - \text{etc.}$$

*Ans.* $\cos 10^\circ = \cdot 9848078 \ldots$

**To find the tangent of 24°**

$$\tan 24^\circ = \frac{2\pi}{15} + \frac{1}{3} \left(\frac{2\pi}{15}\right)^3 + \frac{2}{15} \left(\frac{2\pi}{15}\right)^5 + \text{etc.}$$

*Ans.* $\tan 24^\circ = \cdot 4452287 \ldots$

By formulæ on pages 67, 68, we can find the sine or cosine of any angle less than one radian; but we can also make use of it to find the sine or cosine of an angle greater than one radian.

Suppose we want the sine of 78° Now

$$\begin{aligned}\sin 78^\circ &= \cos 12^\circ \\ &= 1 - \frac{1}{\lfloor 2}\left(\frac{\pi}{15}\right)^2 + \frac{1}{\lfloor 4}\left(\frac{\pi}{15}\right)^4 - \frac{1}{\lfloor 6}\left(\frac{\pi}{15}\right)^6 + \text{etc.} \\ &= \cdot 9781476\end{aligned}$$

*Examples.*

1. Expand $(1+x)e^x$ by Maclaurin's Theorem to six terms.

*Ans.* $1 + 2x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{5x^4}{24} + \frac{x^5}{20} + \text{etc.}$

2. Expand $e^x \sin x$ in powers of $x$

*Ans.* $x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} - \frac{x^7}{630} + \text{etc.}$

3. Expand to five terms $e^x \sec x$

*Ans.* $1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{2} + \text{etc.}$

4. Expand $\log(1 + e^{ax})$ to four terms.

*Ans.* $\text{Log}\, 2 + \frac{ax}{2} + \frac{a^2x^2}{8} - \frac{a^4x^4}{192} \ldots \text{etc.}$

5. Expand $e^{\sin^{-1} x}$ to five terms in powers of $x$

*Ans.* $1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{5x^4}{24} + \text{etc.}$

6. Expand $e^{-x} \log(1+x)$ to four terms in powers of $x$

*Ans.* $x - \frac{3x^2}{2} + \frac{4x^3}{3} - x^4 + \text{etc.}$

7. Prove that $\log\left\{x+\sqrt{1+x^2}\right\} = x - \dfrac{x^3}{6} + \dfrac{3x^5}{40} - \text{etc.}$

8. Expand $(\sec x)^n$ to three terms.

$$\textit{Ans.}\quad 1 + \frac{nx^2}{|\underline{2}} + \frac{(3n^2+2n)x^4}{|\underline{4}} + \text{etc.}$$

9. Expand $e^{a+x} \cos x$ to six terms.

$$\textit{Ans.}\quad e^a\left\{1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \frac{x^5}{30} + \frac{x^7}{630} \cdots \text{etc.}\right\}$$

10. Expand $\sin^{-1}(x+y)$, by Taylor's Theorem, to four terms in powers of $y$

$$\textit{Ans.}\quad \operatorname{Sin}^{-1} x + \frac{y}{\sqrt{1-x^2}} + \frac{y^2}{|\underline{2}}\,\frac{x}{(1-x^2)^{\frac{3}{2}}} + \frac{y^3}{|\underline{3}}\,\frac{1+2x^2}{(1-x^2)^{\frac{5}{2}}} + \text{etc.}$$

# CHAPTER VI

## INDETERMINATE FORMS

An algebraic or transcendental function of a variable is said to be indeterminate when for a particular value of the variable the function assumes one or other of the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0^0$, $\infty^0$ and $1^{\pm\infty}$

For example:

$$\frac{x^2 - ax + cx - ac}{x^2 - ax + bx - ab} = \frac{0}{0} \text{ when } x = a \quad . \quad (a)$$

$$\frac{a^{\frac{1}{\frac{\pi}{2} - x}}}{\tan x} = \frac{\infty}{\infty} \text{ when } x = \frac{\pi}{2} \quad . \quad . \quad (b)$$

$$(x)^{\tan x} = 0^0 \text{ when } x = 0 \quad . \quad . \quad (c)$$

$$(\operatorname{cosec} x)^{\frac{x}{a}} = \infty^0 \text{ when } x = 0 \quad . \quad . \quad (d)$$

$$(\sin x)^{\tan x} = 1^{\infty} \text{ when } x = \frac{\pi}{2} \quad . \quad . \quad (e)$$

$$(1 + x)^{\log x} = 1^{-\infty} \text{ when } x = 0 \quad . \quad . \quad (f)$$

We shall endeavour to show how the true value of such expressions may be obtained when the variable approaches that limit which renders the expression indeterminate.

Expressions such as $(a)$ and $(b)$ are either in the form $\frac{f(x)}{\phi(x)}$ or they can be easily expressed in that form. Now suppose $f(x)$ and $\phi(x)$ are both zero when $x = a$

If we substitute $(x+h)$ instead of $x$ in each function where $h$ is indefinitely small, we have the limiting value of

$$\frac{f(x+h)}{\phi(x+h)} = \frac{f(x)}{\phi(x)}$$

but by a theorem in algebra, each of these fractions is equal to the fraction whose numerator is the difference of the two numerators and whose denominator is the difference of the two denominators, therefore

$$\frac{f(x)}{\phi(x)} = \frac{f(x+h)-f(x)}{\phi(x+h)-\phi(x)} = \frac{\dfrac{f(x+h)-f(x)}{h}}{\dfrac{\phi(x+h)-\phi(x)}{h}}$$

that is $\dfrac{f'(x)}{\phi'(x)}$ and therefore the limiting value of $\dfrac{f(x)}{\phi(x)}$ when $x = a$ is $\dfrac{f'(a)}{\phi'(a)}$

If $\dfrac{f'(x)}{\phi'(x)}$ be indeterminate we proceed to the second derived functions, etc., until one or both functions cease to vanish or become infinite.

**Find the true value of** $\dfrac{x - \sin x}{\sin^3 x}$ when $x = 0$

Here $\dfrac{f(x)}{\phi(x)} = \dfrac{x - \sin x}{\sin^3 x}$ and therefore the limiting value of $\dfrac{x - \sin x}{\sin^3 x}$ when $x = 0$ is $\dfrac{f'(0)}{\phi'(0)}$

that is, $$\frac{1-\cos 0^0}{3\sin^2 0^0 \cos 0^0} = \frac{1-\cos 0^0}{3(1-\cos^2 0^0)\cos 0^0}$$

$$= \frac{1}{3(1+\cos 0^0)\cos 0^0} = \frac{1}{6}$$

**Find the true value of** $\dfrac{\cos px - \cos qx}{\cos mx - \cos nx}$ when $x = 0$

Here $$\frac{f(x)}{\phi(x)} = \frac{\cos px - \cos qx}{\cos mx - \cos nx}$$

the limiting value of which is

$$\frac{-p \sin px + q \sin qx}{-m \sin mx + n \sin nx} = \frac{0}{0}$$

when $x = 0$ which is indeterminate. We proceed to the second derived functions, therefore the limiting value of

$$\frac{-p \sin px + q \sin qx}{-m \sin mx + n \sin nx}$$

is
$$\frac{-p^2 \cos px + q^2 \cos qx}{-m^2 \cos mx + n^2 \cos nx} = \frac{q^2 - p^2}{n^2 - m^2}$$

when $x = 0$

**Find the value of** $x^2 \tan x - \left(\frac{\pi}{2}\right)^2 \sec x$ when $x = \frac{\pi}{2}$

Here we have

$$x^2 \tan x - \left(\frac{\pi}{2}\right)^2 \sec x = \frac{x^2 \sin x - \left(\frac{\pi}{2}\right)^2}{\cos x}$$

the limiting value of which is

$$\frac{2x \sin x + x^2 \cos x}{-\sin x} = -\pi, \text{ when } x = \frac{\pi}{2}$$

**Find the true value of** $(\tan x)^{\sin x}$ when $x = 0$
This is in the form $0^0$

Let $y = (\tan x)^{\sin x}$

$$\therefore \log y = \sin x \log \tan x = \frac{\log \tan x}{\operatorname{cosec} x}$$

the limiting value of which is

$$= -\frac{\sin x}{\cos^2 x} = 0 \text{ when } x = 0$$

that is, $\log y = 0$ when $x = 0$

$\therefore y = 1$

that is, 1 is the limiting value of

$$(\tan x)^{\sin x} \text{ when } x = 0$$

**Find the true value of** $(\tan x)^{\cos x}$ **when** $x = \frac{\pi}{2}$

This is of the form $(\infty)^0$

Let $y = (\tan x)^{\cos x}$

$$\therefore \log y = \cos x \log \tan x = \frac{\log \tan x}{\sec x}$$

the limiting value of which is

$$= \frac{\cot x \sec^2 x}{\sec x \tan x} = \cos x \operatorname{cosec}^2 x = 0 \text{ when } x = \frac{\pi}{2}$$

$$\therefore (\tan x)^{\cos x} = 1 \text{ when } x = \frac{\pi}{2}$$

**Find the true value of** $x^{\frac{1}{1-x}}$ **when** $x = 1$

Let $y = x^{\frac{1}{1-x}}$ $\therefore \log y = \dfrac{1}{1-x} \log x$, the limiting value of which is

$$= \frac{\frac{1}{x}}{-1} = -1 \text{ when } x = 1$$

that is, $\log y = -1$

$\therefore y = e^{-1}$ when $x = 1$

*Examples.*—Find the limiting value of the following expressions.

1. $\dfrac{x^3 - 7x^2 + 4x + 12}{x^2 - 14x + 24}$ when $x = 2$ *Ans.* $\dfrac{6}{5}$

2. $\dfrac{\log x}{x^4 - 1}$ when $x = 1$ *Ans.* $\dfrac{1}{4}$

3. $\dfrac{\sin x - x^2}{x^3}$ when $x = 0$ *Ans.* $\infty$

4. $\dfrac{3e^x - x^3 - 3x - 3}{\tan^2 x}$ when $x = 0$ *Ans.* $\dfrac{3}{2}$

5. $\dfrac{\sin 2\theta + 2\sin^2\theta - 2\sin\theta}{\cos\theta - \cos^2\theta}$ when $\theta = 0$ *Ans.* 4

6. $\dfrac{a^x + b^x - a - b}{\log x}$ when $x = 1$ *Ans.* $\log a^a b^b$

7. $\dfrac{a \sin x - \sin ax}{x(\cos x - \cos ax)}$ when $x = 0$ *Ans.* $\dfrac{a}{3}$

8. $\dfrac{x^3 - 3\cos^2 x + 3}{x^2}$ when $x = 0$ *Ans.* 3

9. $\dfrac{(e^x - e^{-x})^2}{\log(1 + x) - x}$ when $x = 0$ *Ans.* $-8$

10. $\dfrac{\sin^2 4x + 2\cos^2 x - 2\cos x}{\cos^2 x - \cos^3 x}$ when $x = 0$ *Ans.* 30

11. $\dfrac{\log(1 + x^2 + x^4)}{\sec x - \cos x}$ when $x = 0$ *Ans.* 1

12.* $\dfrac{(x + \sin x - 4\sin\frac{1}{2}x)^4}{(3 + \cos x - 4\cos\frac{1}{2}x)^3}$ when $x = 0$ *Ans.* $\dfrac{128}{81}$

13.* $\dfrac{(3\sin x - \sin 3x)^4}{(\sec x - \cos 2x)^6}$ when $x = 0$. *Ans.* 256

14. $\dfrac{e^x - e^a}{\cos\dfrac{\pi x}{2a}}$ when $x = a$ *Ans.* $-\dfrac{2ae^a}{\pi}$

15. $\dfrac{\log x}{\sqrt{1 - x}}$ when $x = 1$ *Ans.* $\infty$

16. $(\cot x)^{\sin ax}$ when $x = 0$ *Ans.* 1

* Expand the sines and cosines before differentiating.

17. $x^{\frac{1}{\log ax}}$ when $x = \infty$ *Ans.* $e$

18. $(1 + ax)^{\frac{1}{x}}$ when $x = 0$ *Ans.* $e^a$

19. $\left(\dfrac{A^{\frac{1}{x}} + B^{\frac{1}{x}} + C^{\frac{1}{x}} + D^{\frac{1}{x}}}{4}\right)^{4x}$ when $x = \infty$ *Ans.* ABCD

20. $\dfrac{\cos 2x + 2\cos^2 x - 2\cos x + 1}{\sin x - \sin^2 x}$ when $x = \dfrac{\pi}{2}$

*Ans.* $\infty$

## CHAPTER VII

# MAXIMA AND MINIMA OF FUNCTIONS OF ONE VARIABLE

WE have already shown in Chapter I how the value of any function of a single variable depends upon the value we assign to the variable.

Now, suppose the variable to increase continuously from one definite value to another, and in consequence of this gradual change, suppose the function to be gradually increasing at one time, and at another time gradually diminishing; there must be some particular value of the variable for which the function ceases to increase and begins to diminish. The corresponding value of the function is called a *maximum value.*

Again, in consequence of this gradual change of the variable, suppose that at one time the value of the function is gradually diminishing, and at another time gradually increasing; there must be some particular value of the variable for which the function ceases to diminish and begins to increase. This value is called a *minimum value of the function.*

The student should observe that the terms maximum and minimum values of a function do not necessarily mean the numerically greatest and least values of the function, as a function may have several maxima and several minima values.

A maximum value of a function may be numerically less than a minimum value of it.

For a particular value of the variable, a function may cease

to increase or diminish; but if it does not begin to diminish or increase respectively on passing through this value, it is neither a maximum nor a minimum.

We shall illustrate the foregoing by means of an example.

Let $y = 12x^5 - 135x^4 + 580x^3 - 1170x^2 + 1080x$ If $x$ increases continuously from zero up to 4, on plotting the values of $x$ horizontally and the corresponding values of $y$ vertically, it will be found that $y = 0$ when $x = 0$, and that as $x$ increases from 0 to 1, the value of $y$ increases until $x = 1$ As $x$ increases from 1 to 2, $y$ diminishes, and when $x = 2$, $y$ ceases to diminish. As

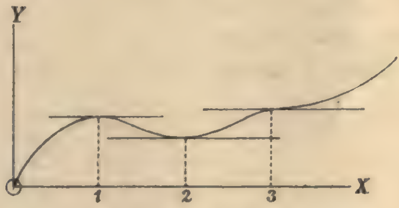


FIG. 26.

$x$ increases from 2 to 3, $y$ increases, and when $x = 3$ the increment of $y$ is zero. As $x$ increases from 3 to 4, $y$ increases: therefore, when $x = 1$, $y$ is a *maximum*, and when $x = 2$, $y$ is a *minimum*, and when $x = 3$, $y$ is *neither* a maximum nor a minimum, since the increment of $y$ does not change in sign as $x$ gets greater than 3.

Now $\frac{dy}{dx}$ represents the rate of increase of the function, or the slope of the curve, as $x$ increases continuously, and may be positive, zero, negative, or infinite, depending upon the instantaneous value of $x$ Now $\frac{dy}{dx}$ will be positive if

function $x$ is approaching a maximum value as $x$ increases, and will be negative if function $x$ is approaching a minimum. When function $x$ is either a maximum or a minimum, $\frac{dy}{dx}$ is zero. This is evident on referring to the figure, since $\frac{dy}{dx}$ represents the tangent of the angle which the tangent to a curve makes with the axis of $x$

Therefore we deduce the following rule for finding the values of $x$ which make function $x$ a maximum or minimum. *Differentiate the function of $x$ and equate the first derived function to zero; the roots of this equation will in general render function $x$ a maximum or minimum.*

It remains to find whether a given root renders function $x$ a maximum, minimum, or neither.

If an adjacent value less than the given root renders $\frac{dy}{dx}$ or first derived function of $x$ $+$, and an adjacent value greater than the root renders $\frac{dy}{dx}$ or first derived function of $x$ $-$; then that root will render function $x$ a *maximum*, since the function is increasing, as $x$ increases, for an adjacent value, and is diminishing, as $x$ increases, for an adjacent greater value.

Similarly, if an adjacent value less than a given root renders first derived function $x$ $-$, and an adjacent greater value than the root renders the first derived function $x$ $+$, then that root will render function $x$ a *minimum*, since the function is diminishing for a smaller, and increasing for a greater value than the given root.

Again, if function $x$ is $+$ or $-$ for an adjacent value less than a given root, and also $+$ or $-$ respectively for an adjacent value greater than the given root, that root will render the function neither a maximum nor minimum, since the first derived function does not change in sign.

If we differentiate the example on page 83 we get

$$\frac{dy}{dx} = 60(x-1)(x-2)(x-3)^2$$

and equating this to zero the roots are 1, 2, and 3.

$$\begin{array}{llll} \text{If } x \text{ be} < & 1 & & + \\ \quad ,, \quad = & 1 & \dfrac{dy}{dx} & 0 \\ \quad ,, \quad > & 1 & & - \end{array}$$

therefore $x = 1$ makes $f(x)$ a maximum.

$$\begin{array}{llll} \text{If } x \text{ be} < & 2 & & - \\ \quad ,, \quad = & 2 & \dfrac{dy}{dx} & 0 \\ \quad ,, \quad > & 2 & & + \end{array}$$

therefore $x = 2$ makes $f(x)$ a minimum.

$$\begin{array}{llll} \text{If } x \text{ be} < & 3 & & + \\ \quad ,, \quad = & 3 & \dfrac{dy}{dx} & 0 \\ \quad ,, \quad > & 3 & & + \end{array}$$

therefore $x = 3$ makes $f(x)$ neither a maximum nor minimum.

Differentiate the first derived function;

$$\therefore \frac{d^2y}{dx^2} = 60(4x^3 - 27x^2 + 58x - 39)$$

Now $\frac{d^2y}{dx^2}$ represents the rate of change of the slope of a curve at any point on the curve; therefore, when the function is a maximum the change of slope will be negative, and when the function is a minimum the change of slope will be positive, as $x$ gradually increases: therefore *that root which renders* $\frac{d^2y}{dx^2}$ *negative will render function* $x$ *a maximum, and that root which renders* $\frac{d^2y}{dx^2}$ *positive will make function* $x$ *a minimum, and that root which renders* $\frac{d^2y}{dx^2}$

*zero will in general render function $x$ neither a maximum nor a minimum.*

Substituting 1, 2, and 3 for $x$, in $4x^3 - 27x^2 + 58x - 39$ it will be $-$, $+$ and 0 respectively; therefore $x = 1$ a maximum, $x = 2$ a minimum, $x = 3$ neither.

We have stated that a value of $x$ which renders $\frac{d^2y}{dx^2}$ zero will in general render function $x$ neither a maximum nor a minimum. That value of $x$ which renders $\frac{d^2y}{dx^2}$ zero *may* render function $x$ a maximum or a minimum, as we shall now endeavour to show.

Let $a$ be that value of $x$ which renders $f(x)$ a maximum or a minimum; then

$$f(a+h) - f(a) \quad \text{and} \quad f(a-h) - f(a)$$

will both be negative if $f(a)$ be a maximum, and will both be positive if $f(a)$ be a minimum.

By **Taylor's Theorem** we have

$$f(a+h) - f(a) = hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a) + \text{etc.} \quad (\alpha)$$

$$f(a-h) - f(a) = -hf'(a) + \frac{h^2}{\lfloor 3}f''(a) - \frac{h^3}{\lfloor 3}f'''(a) + \text{etc.} \quad (\beta)$$

In these equations ($\alpha$) and ($\beta$), if $h$ be very small, it is evident that the sign of the right-hand side is the same as the sign of the first term; therefore the condition for either a maximum or minimum is that $f'(a) = 0$, for if not, then $f(a+h) - f(a)$ and $f(a-h) - f(a)$ would be opposite in sign.

$$\therefore f(a+h) - f(a) = \frac{h^2}{\lfloor 2}f''(a) + \frac{h^3}{\lfloor 3}f'''(a) + \frac{h^4}{\lfloor 4}f''''(a) + \text{etc.}$$

and

$$f(a-h)-f(a) = \frac{h^2}{\underline{|2}}f''(a) - \frac{h^3}{\underline{|3}}f'''(a) + \frac{h^4}{\underline{|4}}f''''(a) - \text{etc.}$$

If $f(a)$ be a maximum, $f(a \pm h) - f(a)$ must be negative; therefore $f''(a)$ must be negative if it be not zero, since $h^2$ is positive. If $f(a)$ be a minimum, $f(a \pm h) - f(a)$ must be positive. Hence for a maximum or minimum $f''(a)$ must be negative or positive respectively, if it be not zero.

Again, if $f''(a)$ be zero, then the condition for a maximum or minimum is that $f'''(a) = 0$ otherwise $f(a+h) - f(a)$ and $f(a-h) - fa$ would be opposite in sign; therefore for a maximum $f''''(a)$ must be negative if not zero and for a minimum $f''''(a)$ must be positive if not zero.

By similar reasoning it can be shown that if $n$ be odd, if the first $n$ differential coefficients of $f(x)$ vanish when $x = a$ then $f(x)$ is a maximum or minimum according as the $(n+1)$th differential coefficient is negative or positive, and if $n$ be even, $f(a)$ is neither a maximum nor a minimum.

**To find the values of $x$ which will make $y = 3x^4 - 8x^3 - 18x^2 + 72$ a maximum or minimum.**

Here $\dfrac{dy}{dx} = 12(x^3 - 2x^2 - 3x) = 0$ for a max. or min.

$$\therefore x = -1,\ 0,\ \text{or } 3$$

$$\frac{d^2y}{dx^2} = 36x^2 - 48x - 36$$

If $x = -1$ $\dfrac{d^2y}{dx^2} = +48$ positive. $\therefore x = -1$, a min.

If $x = 0$ $\dfrac{d^2y}{dx^2} = -36$ negative. $\therefore x = 0$, a max.

If $x = 3$ $\dfrac{d^2y}{dx^2} = +144$ positive. $\therefore x = 3$, a min.

**To find the maximum and minimum values of** $y = \dfrac{x^2 - 7x + 6}{x - 10}$

$$\frac{dy}{dx} = \frac{(x-4)(x-16)}{(x-10)^2} = 0 \text{ for a max. or min.}$$

When $x < 4$, $= 4$, $> 4$, $\frac{dy}{dx}$ $+$, $0$, $-$ respectively, therefore $x = 4$, a maximum.

When $x < 10$, $= 10$, $> 10$, $\frac{dy}{dx}$ $-$, $\infty$, $-$, therefore $x = 10$ neither a max. nor min., since $\frac{dy}{dx}$ does not change in sign.

When $x < 16$, $= 16$, $> 16$, $\frac{dy}{dx}$ $-$, $0$, $+$ respectively, therefore $x = 16$, a minimum.

If $x = 4$, $y = 1$, a maximum. If $x = 16$, $y = 25$, a minimum.

In this example the maximum is less than the minimum.

It would be interesting to plot on squared paper the curve representing the relation between $y$ and $x$, observing what occurs as $x$ gradually increases from 0 up to 20.

**Divide a line into two parts such that the rectangle under them may be a maximum.**

Let $a$ denote the line. Then, if $x$ denote one part, $a - x$ will denote the other part, and $x(a - x)$ is to be a maximum.

Let $y = x(a - x)$ Therefore $\frac{dy}{dx} = a - 2x = 0$ for a maximum.

$$\therefore \ x = \frac{a}{2} \text{ that is, the line must be bisected.}$$

**Find two factors of $a$, so that the sum of their squares is a minimum.**

Let $x$ denote one factor, then $\frac{a}{x}$ denotes the other factor. Therefore $y = x^2 + \frac{a^2}{x^2}$ is to be the minimum.

$\frac{dy}{dx} = 2x - \frac{2a^2}{x^3} = 0$ for a minimum. $\therefore x = \sqrt{a}$

**The strength of a rectangular beam of given length and material and loaded in any particular way, is proportional to its breadth and to the square of its depth. What is the breadth of the strongest beam that can be cut from a cylindric tree of 12 inches diameter?**

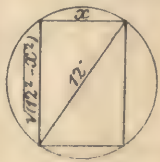

FIG. 27.

Let $x$ denote the required breadth in inches; then $\sqrt{12^2 - x^2}$ will denote the depth. The strength is proportional to $x(12^2 - x^2)$

Let $y = x(12^2 - x^2)$

$$\therefore \frac{dy}{dx} = 144 - 3x^2 = 0 \text{ for a maximum.}$$

$$\therefore x = 4\sqrt{3} \text{ inches, and depth } = 4\sqrt{6} \text{ inches.}$$

**To find the depth of the stiffest beam that can be cut from a cylindric tree 12 inches in diameter and of given length. Stiffness is proportional to the breadth and to the cube of the depth.**

Let $x$ denote the required breadth; then $\sqrt{12^2 - x^2}$ will denote the depth in inches.

The stiffness is proportional to $x(12^2 - x^2)^{\frac{3}{2}}$

Let $y = x(12^2 - x^2)^{\frac{3}{2}}$

$\therefore \frac{dy}{dx} = (12^2 - x^2)^{\frac{3}{2}} - 3x^2(12^2 - x^2)^{\frac{1}{2}} = 0$ for a maximum.

Therefore $144 - 4x^2 = 0 \quad \therefore x = 6$ inches, the required breadth; therefore the required depth is $6\sqrt{3}$ inches.

**If $I^2r + \frac{t^2}{r}$ is the total waste per mile going on in an electric conductor ($r$ ohms resistance per mile), due to heat, interest and depreciation, find the relation between I, $r$ and $t$ when the waste is a minimum.**

Let $y = I^2r + \frac{t^2}{r}$, I and $t$ being constants

$$\frac{dy}{dr} = I^2 - \frac{t^2}{r^2} = 0 \text{ for a minimum}$$

Therefore $Ir = t$

In Lord Kelvin's rule let $Ir = 17$, where I is the current in amperes, $r$ the resistance in ohms per mile of conductor. The numerical value of $r$ in terms of the cross-sectional area $a$ of conductor is $\frac{\cdot 04}{a}$ approximately.

$$\therefore a = \frac{I}{425}$$

For a minimum cost the carriage is therefore 425 amperes per square inch of cross-sectional area of conductor.

**To find the proper section of a conductor to transmit a given power of P watts over a distance of $n$ miles, taking into account the drop in potential due to the distance.**

Let $V_{,}$ be the potential at the generator, and V the potential at the motor, $r$ the resistance in ohms per mile, therefore the total resistance is $nr$ and $V_{,} - V =$ drop in volts $= nrI$ where I is the current in amperes; therefore

$$V = V_{,} - nrI \quad \text{and} \quad P = I(V_{,} - nrI);$$

therefore
$$r = \frac{IV_{,} - P}{nI^2}$$

Substituting this value of $r$ in the rule for waste,

$$I^2 r + \frac{t^2}{r}$$

we have waste

$$W = \frac{I^2(IV_{,} - P)}{nI^2} + \frac{t^2 n I^2}{IV_{,} - P}$$

We want the waste to be a minimum, C being the independent variable, $V_{,}$, $P$, $n$, and $t$ being constants,

$$\therefore \frac{dW}{dI} = \frac{V_{,}}{n} + \frac{2nt^2 I(IV_{,} - P) - V_{,} t^2 n I^2}{(IV_{,} - P)^2} = 0$$

for a minimum, therefore

$$V_{,}(IV_{,} - P)^2 + 2n^2 t^2 I(IV_{,} - P) - V_{,} t^2 n^2 I^2 = 0 \quad (a)$$

We add a numerical example for the student to work out.

Let

$P = 120000$, $n = 5$, $V_{,} = 2000$ and $t = 17$

Substituting these values in $(a)$, we get I, and $a = \frac{I}{425}$, which gives the cross-section of the conductor.

**If $v$ be the velocity of an ocean current in knots, $x$ the velocity of a ship through the water in knots, and if the quantity of fuel burnt per hour be proportional to $x^3$; find the velocity of the ship so as to make the consumption of fuel a minimum for any given distance traversed.**

The velocity of the ship relatively to the still water is $x - v$ the time occupied by a journey of $s$ miles is $\frac{s}{x - v}$ the fuel burnt per hour is $cx^3$ where $c$ is a constant.

$$\therefore \text{Total fuel} = \frac{cx^3 s}{x - v} \quad \text{or} \quad a\frac{x^3}{x - v}$$

hence $\frac{x^3}{x - v}$ is to be a minimum

Let $y = \dfrac{x^3}{x - v}$ $\therefore \dfrac{dy}{dx} = \dfrac{3x^2(x - v) - x^3}{(x - v)^2} = 0$ for a minimum. This gives $x = \frac{3}{2}v$

**Given $n$ voltaic cells of E.M.F. $e$, and internal resistance $r$, to find the way in which they should be arranged to send a maximum current through a given external resistance R.**

Let $x$ cells be placed in series, therefore $\dfrac{n}{x}$ will be in parallel. The total E.M.F. $= xe$ and the total resistance $= \dfrac{x^2r}{n} + \mathrm{R}$; therefore the current $\mathrm{I} = \dfrac{xe}{\dfrac{x^2r}{n} + \mathrm{R}}$

$$\therefore \frac{d\mathrm{I}}{dx} = \frac{e\left(\dfrac{x^2r}{n} + \mathrm{R}\right) - xe\left(\dfrac{2xr}{n}\right)}{\left(\dfrac{x^2r}{n} + \mathrm{R}\right)^2} = 0 \text{ for a maximum}$$

This gives $\mathrm{R} = \dfrac{x^2r}{n}$; that is, arrange them so that the external resistance may be equal to the internal resistance.

**A man is at sea 4 miles distant from the nearest point on the land, and he wishes to get to a place 10 miles distant from the nearest point, the road lying along the shore and being straight, so that he can row or walk. Find at what point he must land in order to get to this place in a minimum time. He rows at 3 miles per hour and walks at 4 miles per hour.**

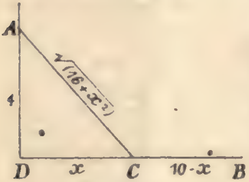


FIG. 28.

Let $x$ denote the distance in miles from the nearest point, D, to C, the point where he must land; then $10 - x$ will denote the distance in miles he must walk.

The time in hours $t = \dfrac{\sqrt{16+x^2}}{3} + \dfrac{10-x}{4}$

$$\therefore \frac{dt}{dx} = \frac{x}{3\sqrt{16+x^2}} - \frac{1}{4} = 0 \text{ for a minimum}$$

$$\therefore 16x^2 = 9(16+x^2) \qquad \therefore x = \tfrac{12}{7}\sqrt{7} \text{ miles}$$

$$\therefore t = 3{\cdot}382 \text{ hrs.}$$

**To find the greatest cone that can be inscribed in a sphere of given radius** $r$

Let the height of the cone be $x + r$; then the radius of its base is $\sqrt{r^2 - x^2}$, and its volume

$$V = \frac{\pi}{3}(x+r)(r^2 - x^2)$$

$$\therefore \frac{dV}{dx} = \frac{\pi}{3}(r^2 - 2rx - 3x^2) = 0 \text{ for a maximum}$$

$\therefore x = \dfrac{r}{3}$ therefore the height is $\dfrac{4}{3}r$ and the volume $= \dfrac{32\pi r^3}{81}$

**To find the length of the arc of a sector which must be cut from a circular piece of sheet iron so that the remainder may form a conical vessel of maximum capacity.**

Let $r$ denote the radius of the circular piece, and let $x$ denote the semi-vertical angle of the conical vessel. Its height will therefore be $r\cos x$, the radius of its base $r\sin x$, and its volume $V = \dfrac{\pi}{3}r^3 \sin^2 x \cos x$

Let $y = \sin^2 x \cos x \quad \therefore \dfrac{dy}{dx} = 2\sin x\cos^2 x - \sin^3 x = 0$ for a maximum. $\therefore \tan x = \sqrt{2}$

$\therefore \sin x = \sqrt{\tfrac{2}{3}}$ and the circumference of its base is $2\pi r\sqrt{\tfrac{2}{3}}$ therefore the arc of the sector $= 2\pi r(1 - \sqrt{\tfrac{2}{3}})$

*Examples.*

1. $y = 4x^3 - 3x^2 - 18x$

   *Ans.* $\begin{cases} x = -1, \text{ a maximum} \\ x = \frac{3}{2}, \text{ a minimum} \end{cases}$

2. $y = 2x^3 - 9x^2 + 12x - 4$

   *Ans.* $\begin{cases} x = 1, \text{ a maximum} \\ x = 2, \text{ a minimum} \end{cases}$

3. $y = \dfrac{x^2 - 7x + 6}{x - 10}$

   *Ans.* $\begin{cases} x = 4, \text{ a maximum} \\ x = 16, \text{ a minimum} \end{cases}$

4. $y = \dfrac{3x}{9 + x^2}$

   *Ans.* $\begin{cases} x = 3, \text{ a maximum} \\ x = -3, \text{ a minimum} \end{cases}$

5. $y = \dfrac{\log x}{x}$

   *Ans.* $x = e$, a maximum

6. $y = x^{\frac{1}{x}}$

   *Ans.* $x = e$, a maximum

7. $y = ae^{bx} + ce^{-bx}$

   *Ans.* $x = \dfrac{1}{2b} \log \left(\dfrac{c}{a}\right)$, a minimum

8. $y = \dfrac{(a + x)(b + x)}{x}$

   *Ans.* $x = \sqrt{ab}$, a minimum

9. Two trains are running uniformly at the rate of 30 miles and 40 miles an hour along lines at right angles to one another. Show that if their distances at one time from the point of crossing of the lines be 30 miles and 20 miles respectively, the least distance between them is 12 miles.

10. Find the inclination of a smooth plane so that a body sliding down it may pass over a given horizontal distance in the least possible time.

*Ans.* $45°$

11. $y = \sin^6 x \cos x$

*Ans.* $\tan x = \sqrt{6}$, a maximum

12. $y = \dfrac{\sin^3 x}{1 - \cos x}$

*Ans.* $x = \dfrac{\pi}{3}$, a maximum

13. $y = \dfrac{2}{1 + x - x^2}$

*Ans.* $x = \frac{1}{2}$, a minimum

14. Find the cone of maximum volume that can be inscribed in a hemisphere of radius $r$, its vertex being at the centre of the sphere.

*Ans.* $\text{Vol} = \dfrac{2\sqrt{3}\pi r^3}{27}$

15. Find the cone of minimum volume that can be described about a given hemisphere.

*Ans.* $\text{Vol} = \dfrac{\pi r^3\sqrt{3}}{2}$

16. The area of a rectangle is given; find the ratio of the lengths of two adjacent sides if the sum of three sides is a minimum.

*Ans.* $2 : 1$

17. Find the cone of minimum surface, including the base, that can be described about a given hemisphere.

*Ans.* $\text{Vol} = \dfrac{8\pi r^3}{9}$

18. Find the height of the flame of a lamp standing on the centre of a round table 4 feet in diameter, so that a given

horizontal area at its edge may receive the greatest illumination from it. The intensity of light varies directly as the sine of the angle which a ray makes with the plane of the table, and inversely as the square of the distance.

*Ans.* Height $= 12\sqrt{2}$ inches

19. Find the greatest cylinder that can be inscribed in a given hemisphere.

*Ans.* Vol $= \dfrac{2\sqrt{3}\pi r^3}{9}$

20. A line AB is terminated by the axes of X and Y and passes through a fixed point P, whose distances from the axes of X and Y are 5 and 3 respectively; find its direction when $AP^2 + BP^2$ is a minimum.

*Ans.* $\text{Tan}^{-1} - \sqrt{\dfrac{5}{3}}$ with axis of X

21. Find the dimensions of the strongest rectangular beam of given length that can be cut from a cylindric tree of $a$ feet in diameter.

*Ans.* Breadth $= \dfrac{a\sqrt{3}}{3}$

Depth $= \dfrac{a\sqrt{6}}{3}$

22. Find the stiffest beam that can be cut from the tree mentioned in Example 21.

*Ans.* Depth $= \dfrac{a\sqrt{3}}{2}$

Breadth $= \dfrac{a}{2}$

23. Find the proper section of a conductor to transmit 60,000 watts over a distance of 10 miles, the potential at the generator being 2000 volts.

*Ans.* ·076 sq. inch

24. At what distance from the wall of a house must a man, whose eye is $5\frac{1}{2}$ feet from the ground, station himself, in order that a window 5 feet high, whose sill is $20\frac{1}{2}$ feet above the ground, may subtend the greatest vertical angle?

*Ans.* $10\sqrt{3}$ feet.

25. Given the length of an arc of a circle $l$, find what portion of a circle it must be so that the corresponding segment shall be a maximum.

*Ans.* $l = \pi r$ where $r$ is the radius of the circle.

26. A beam of length $l$ feet is supported at the ends and loaded uniformly with $\omega$ tons per foot-run; find where the bending moment is a maximum.

*Ans.* At the centre.

27. If the beam in the preceding example be 12 feet long and loaded with 2 tons per foot-run, and also loaded at 4 feet from the right-hand support with a concentrated load of 2 tons; find where the bending moment is a maximum.

*Ans.* 6 feet 8 inches from the left-hand support.

28. A cylindrical tank closed at both ends is required to hold 40,000 gallons of petrol; find its dimensions if the material used is to be a minimum.

*Ans.* Height is equal to the diameter $= 20{\cdot}13$ feet.

29. A long strip of paper 12 inches broad is folded over as shown in the figure. Find the minimum area of the triangle ABC.

*Ans.* Area $= 32\sqrt{3}$ square inches.
The angle CAB $= 60°$

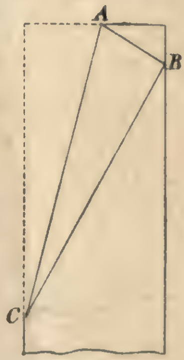


Fig. 29.

30. Find the maximum parcel which can be sent by post subject to the condition that the length

and girth must not exceed 6 feet, (1) when parcel is rectangular, (2) when it is cylindrical.

*Ans.* (1) Length 2 feet. Ends square side 1 foot.
(2) Radius $=$ ·635 foot nearly. Length 2 feet.

31. Suppose the speed of signalling in a long cable varies as $x^2 \log \frac{1}{x}$ where $x$ is the ratio of the diameter of the core to the diameter of the cover; find $x$ when the speed is a maximum.

*Ans.* $x = \dfrac{1}{\sqrt{e}}$

32. In a series-wound dynamo, given $V = \dfrac{aI}{1+nI}$, $I = \dfrac{V}{R+r}$, find R when the power P is a maximum.

$$\textit{Ans.}\quad R = \frac{-(a+2r)+\sqrt{(a+2r)^2-4r(a-r)}}{2}$$

$$\text{and}\quad P = I^2R = \frac{R}{n^2}\left\{\frac{a}{R+r}-1\right\}^2$$

Find $\dfrac{dP}{dR}$ and equate to zero and solve for R

33. Find the cone of minimum volume which can be described about a sphere.

*Ans.* Volume $=$ Twice vol. of sphere.

34. The range of a projectile on a horizontal plane is given by $R = \dfrac{2V^2 \sin\theta \cos\theta}{g}$; find $\theta$ when R is a maximum.

*Ans.* $\theta = \dfrac{\pi}{4}$

35. A rectangular sheet of cardboard 20 inches long and 12 inches broad has equal squares cut out of the corners.

and is then folded up so as to form a box; find the side of the squares when the volume of the box is a maximum.

*Ans.* Side of square $=$ 2·42 inches.

36. Find the maximum rectangle which can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

*Ans.* $2ab$

37. Two spheres whose radii are 25 feet and 16 feet are 100 feet apart from centre to centre; find at what point, on the line joining their centres, there is the maximum amount of their surfaces visible.

*Ans.* At $44\frac{4}{9}$ feet from the centre of the smaller.

38. Find the maximum cone which can be inscribed in the paraboloid of revolution generated by the revolution of the parabola $y^2 = 4ax$ about the axis of $x$ the vertex of the cone being on the axis of $x$ at a distance $d$ from the vertex.

*Ans.* Vol $= \frac{\pi a d^2}{3}$

39. Assuming that the mean pressure of the steam in the cylinder of an engine is given by $a - bn$ where $n$ is the speed, find the speed at which the horse-power is a maximum.

*Ans.* $n = \frac{a}{2b}$

40. Find the maximum value of $x^3e^{-x}$

*Ans.* $27e^{-3}$

41. Assuming that the dynamical load on the connecting rod of an engine is directly proportional to the distance from the cross-head, find where the bending moment is a maximum.

*Ans.* $\cdot 577l$ from the cross-head, $l$ being the length.

42. Assuming that the quantity of water flowing through a submerged broad-crested weir is given by

$$Q = kh\sqrt{H-h}$$

where H is the total head above the crest and $h$ is the depth over the crest, find $h$ in terms of H when the flow is a maximum.

*Ans.* $h = \frac{2}{3}H$

43. Find the ratio of the height to the diameter of a cone of given volume when the curved surface is a minimum.

*Ans.* $h : d :: 2\sqrt{2} : 1$

## CHAPTER VIII

# DIFFERENTIATION OF A FUNCTION OF TWO VARIABLES

LET $u = f(x, y)$

It is required to find $du$ where $x$ and $y$ are both variable. Suppose $y$ to be constant and $x$ to receive a small increment, then

$$du = \left(\frac{du}{dx}\right)dx$$

Again, suppose $x$ constant and $y$ to vary, then

$$du = \left(\frac{du}{dy}\right)dy$$

It is obvious that the total increment of $u$ is the sum of the increments due to both $x$ and $y$ being increased; therefore

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy$$

$\left(\frac{du}{dx}\right)$ is called the partial differential coefficient of $u$ with regard to $x$, and $\left(\frac{du}{dy}\right)$ is the partial differential coefficient of $u$ with regard to $y$

Similarly, if $u = f(x, y, z)$

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy + \left(\frac{du}{dz}\right)dz$$

If $u$ or $f(x, y)$ be constant, then

$$\left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy = 0$$

$$\therefore \frac{dy}{dx} = -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dy}\right)}$$

*Example.*—Let $u = ax^2 + 2hxy + by^2$

Here $\frac{du}{dx} = 2ax + 2hy$ treating $y$ as a constant.

$$\therefore du = 2(ax + hy)dx$$

Similarly, $du = 2(hx + by)dy$ where $x$ is constant,

$$\therefore du = 2(ax + hy)dx + 2(hx + by)dy$$

The relation between the pressure, P, absolute temperature, T, and volume, V, of a gas is given by the formula

$$PV = RT$$

where R is a constant. This relation can be written in the form $V = \frac{RT}{P}$

$$\therefore \delta V = R\frac{\delta T}{P} - \frac{RT\delta P}{P^2}$$

Taking P in lbs. per square foot and V the volume of one pound of air, the value of R is 96 approximately.

The change in volume due to the temperature changing from 273° to 274° C., and the pressure changing from 2116 to 2115 lbs. per square foot, is given by

$$\delta V = \frac{96 \times 1}{2116} - \frac{96 \times 273}{2115^2} = \cdot 0094 \text{ cubic foot}$$

Given $P = \frac{RT}{V}$

then $\delta P = \frac{R\delta T}{V} - \frac{RT\delta V}{V^2} = \frac{R\delta T}{V} - \frac{P}{V}\delta V$

If T changes from 273° to 274° and V changes from 12·4 to 12·5

then $\delta P = \dfrac{96 \times 1}{12{\cdot}4} - \dfrac{2116 \times {\cdot}1}{12{\cdot}5} = -9{\cdot}3$ lbs. per sq. ft.

Given $u = f(x + at) + F(x - at)$, find the relation between $\dfrac{d^2u}{dt^2}$ and $\dfrac{d^2u}{dx^2}$

$$\frac{du}{dt} = af'(x + at) - aF'(x - at)$$

and
$$\frac{d^2u}{dt^2} = a^2f''(x + at) + a^2F''(x - at)$$

Similarly
$$\frac{d^2u}{dx^2} = f''(x + at) + F''(x - at)$$

$$\therefore \frac{d^2u}{dt^2} = a^2\frac{d^2u}{dx^2}$$

This is the differential equation for the small transverse vibration of an elastic string in tension.

Given
$$v = e^{-k^2m^2t} \sin mx$$

show that
$$\frac{dv}{dt} = k^2\frac{d^2v}{dx^2}$$

We have
$$\frac{dv}{dt} = -k^2m^2e^{-k^2m^2t} \sin mx$$

Also
$$\frac{dv}{dx} = me^{-k^2m^2t} \cos mx$$

and
$$\frac{d^2v}{dx^2} = -m^2e^{-k^2m^2t} \sin mx$$

$$\therefore \frac{dv}{dt} = k^2\frac{d^2v}{dx^2}$$

This is the differential equation for the linear flow of heat across a long plate when the edge is suddenly raised in temperature.

Given
$$x = R(\theta - \sin \theta)$$

and
$$y = R(1 - \cos \theta), \text{ find } \frac{dy}{dx}$$

Here $\qquad dy = \text{R} \sin \theta d\theta$

and $\qquad dx = \text{R}(1 - \cos \theta)d\theta$

$$\therefore \frac{dy}{dx} = \frac{\sin \theta}{1 - \cos \theta} = \cot \tfrac{1}{2}\theta$$

Given $v = \dfrac{1}{(x^2 + y^2 + z^2)^{\frac{1}{2}}}$

show that $\qquad \dfrac{d^2v}{dx^2} + \dfrac{d^2v}{dy^2} + \dfrac{d^2v}{dz^2} = 0$

Given $x^{ay} - y^{bx} = 0$ find $\dfrac{dy}{dx}$

Here $\qquad ay \log x = bx \log y$

therefore $\qquad a\dfrac{dy}{dx} \log x + \dfrac{ay}{x} = b \log y + \dfrac{bx}{y}\dfrac{dy}{dx}$

$$\therefore \frac{dy}{dx}\left(a \log x - \frac{bx}{y}\right) = b \log y - \frac{ay}{x}$$

$$\therefore \frac{dy}{dx} = \frac{bxy \log y - ay^2}{axy \log x - bx^2}$$

*Examples.*

1. If $v = \dfrac{z - y}{z + y}$

prove that $\qquad dv = \dfrac{2(ydz - zdy)}{(y + z)^2}$

2. If $v = \dfrac{x^a}{y^b}$

prove that $\qquad dv = \dfrac{x^{a-1}}{y^{b+1}}(aydx - bxdy)$

3. If $v = \tan^{-1}\left(\dfrac{z}{y}\right)$

prove that $\qquad dv = \dfrac{ydz - zdy}{y^2 + z^2}$

4. If $v = a^{xyz}$

prove that $\qquad dv = a^{xyz} \log a \,(xydz + xzdy + yzdx)$

5. If $v = a^{x+y+z}$

prove that $dv = x^{x+y+z} \log a(dx + dy + dz)$

6. If $v = \sec^{-1}\left(\frac{z}{y}\right)$ where $y = \phi(x)$ and $z = f(x)$

prove that $$\frac{dv}{dx} = \frac{1}{\sqrt{(z^2 - y^2)}}\left\{\frac{\phi(x)f'(x) - f(x)\phi'(x)}{f(x)}\right\}$$

7. If $v = \sqrt{\frac{x^2 - y^2}{x^2 + y^2}}$

prove that $$dv = \frac{2xy(ydx - xdy)}{(x^2 + y^2)^{\frac{3}{2}}\sqrt{(x^2 - y^2)}}$$

8. If $v = \frac{x^2 y}{a^2 - z^2}$

prove that $$dv = \frac{2xydx + x^2dy}{a^2 - z^2} + \frac{2x^2yzdz}{a(^2 - z^2)^2}$$

9. If $v = \log \tan\left(\frac{x}{y}\right)$

prove that $$dv = \frac{2ydx - 2xdy}{y^2 \sin \frac{2x}{y}}$$

10. If $u = x^3 + y^3 + 3x^2y + 4xy^2$

prove that $$\frac{d^2u}{dydx} = \frac{d^2u}{dxdy}$$

11. Given $u = ax^3 + bx^2yz + cxy^2 + dz^3$

find $\frac{du}{dx}$, $\frac{du}{dy}$ and $\frac{du}{dz}$

$$\frac{du}{dx} = 3ax^2 + 2bxyz + cy^2$$

$$\frac{du}{dy} = bx^2z + 2cxy$$

$$\frac{du}{dz} = bx^2y + 3dz^2$$

CHAPTER IX

# TANGENTS AND NORMALS TO PLANE CURVES

LET P and Q be two points on a plane curve A B and suppose the point Q to move up indefinitely near to P, then the straight line drawn through P and Q when indefinitely near, is called a *tangent* to the curve at the point P

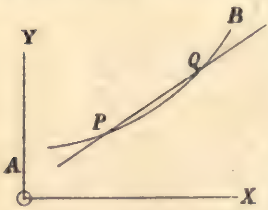

FIG. 30.

Let $(x', y')$ and $(x'', y'')$ be the co-ordinates of P and Q respectively; then the equation of the straight line passing through P and Q is

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x')$$

When P and Q are indefinitely near $y'' - y' = dy$ and $x'' - x' = dx$ therefore the equation of the tangent to a curve at a point $(x', y')$ is

$$y - y' = \frac{dy}{dx}(x - x') \quad . \quad . \quad . \quad . \quad (a)$$

**To find the equation of the tangent to the curve whose equation is $x^2 + y^2 = r^2$, at the point $(x', y')$.**

From the equation to the curve we have

$$\frac{dy}{dx} = -\frac{x}{y} = -\frac{x'}{y'}$$

at the point $(x', y')$

Substituting in ($a$) for $\dfrac{dy}{dx}$ we get

$$y - y' = -\frac{x'}{y'}(x - x')$$

that is,

$$yy' + xx' = x'^2 + y'^2 = r^2$$

Therefore the equation of the tangent to the curve $x^2 + y^2 = r^2$ at the point $(x', y')$ is

$$xx' + yy' = r^2$$

The *normal* to a curve at the point $(x', y')$ is the straight line drawn through the point perpendicular to the tangent at that point. Its equation is therefore

$$y - y' = -\frac{dx}{dy}(x - x') \quad . \quad . \quad . \quad . \quad (\beta)$$

**To find the equation of the normal to the curve $x^2 + y^2 = r^2$ at the point $(x', y')$**

From the last example we have

$$\frac{dy}{dx} = -\frac{x'}{y'} \text{ at the point } (x', y')$$

and on substituting in ($\beta$) we have

$$y - y' = \frac{y'}{x'}(x - x')$$

which reduces to $xy' - x'y = 0$

**To find the equations of the tangent and normal to the curve whose equation is**

$$ax^2 + 2hxy + by^2 = k$$

at the point $(x', y')$

$$\frac{dy}{dx} = -\frac{\dfrac{du}{dx}}{\dfrac{du}{dy}}$$

Substituting for $\frac{dy}{dx}$ in ($\alpha$) we have, for the equation of the tangent,

$$(y-y')\frac{du}{dy}+(x-x')\frac{du}{dx} = 0 \quad . \quad . \quad . \quad (\delta)$$

Now,

$$\frac{du}{dy} = 2(hx'+by') \quad \text{and} \quad \frac{du}{dx} = 2(ax'+hy')$$

therefore the equation of the tangent at the point $(x', y')$ is

$$(ax'+hy')x+(hx'+by')y = k$$

The equation of the normal at the point $(x', y')$ is

$$(y-y')\frac{du}{dx}-(x-x')\frac{du}{dy} = 0$$

that is,

$$(hx'+by')x-(ax'+hy')y = h(x'^2-y'^2)-(a-b)x'y'$$

**To find the tangent and normal to the curve $x^n y^m = c$ at the point $(x', y')$**

Here we have $n \log x + m \log y = \log c$

therefore $$\frac{ndx}{x}+\frac{mdy}{y} = 0$$

that is, $$\frac{dy}{dx} = -\frac{ny'}{mx'}$$ at the point $(x', y')$

On substituting in the formula ($\alpha$) we get

$$\frac{nx}{x'}+\frac{my}{y'} = m+n$$

The normal is given by formula ($\beta$) which leads to

$$\frac{mx}{y'}-\frac{ny}{x'} = \frac{mx'}{y'}-\frac{ny'}{x'}$$

## Subtangents and Subnormals.

Let EPT represent the tangent to the curve AB at the point P, then PF is the normal and PD the perpendicular on the axis of X

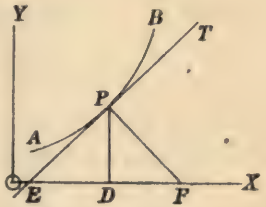


FIG. 31.

Then ED is called the *subtangent*, and DF the *subnormal*.

Denoting the angle PED by $\theta$ = DPF we have ED the subtangent = DP $\cot \theta$

$$\therefore \textbf{ the subtangent } = y\frac{dx}{dy} \quad . \quad . \quad (\alpha)$$

and

$$\text{DF } \textbf{the subnormal} = y\frac{dy}{dx} \quad . \quad . \quad (\beta)$$

**Find the subtangent and subnormal to the curve** $y^2 = 4ax$ at the point $(x', y')$

Here $\dfrac{dy}{dx} = \dfrac{2a}{y} = \dfrac{y'}{2x'}$ at the point $(x', y')$

Substituting in $(\alpha)$ we get

$$\textit{Subtangent} = y' \times \frac{2x'}{y'} = 2x'$$

that is, the subtangent for this curve, which is a parabola, is always double the abscissa.

$$\text{The } \textit{subnormal} = y' \times \frac{y'}{2x'} = \frac{y'^2}{2x'} = 2a$$

that is, the subnormal is constant and equal to twice the distance of the focus from the vertex.

**Find the subtangent and subnormal to the curve whose equation is** $y = e^{ax}$

Here $\frac{dy}{dx} = ae^{ax}$

Therefore *Subtangent* $= e^{ax} \times \frac{1}{ae^{ax}} = \frac{1}{a}$

and the *Subnormal* $= e^{ax} \times ae^{ax} = ae^{2ax}$

**Find the subtangent and subnormal to the catenary whose equation is**

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$$

Here $\frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}$

Therefore the *Subtangent* $= a\frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}$

and *Subnormal* $= \frac{a}{4}\left\{e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}\right\}$

In Fig. 31, PE is called the *length of the tangent* and PF the *length of the normal.*

Now, PE $= y \operatorname{cosec} \theta = y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

$\therefore$ **Tangent** $= y\sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}$

And PF the **Normal** $= y\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}$

**Find the tangent and normal to the catenary,**

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$$

Here $$\frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}$$

therefore **Tangent** $$= \frac{a}{2}\frac{\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}^2}{\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}}$$

**Normal** $$= \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}\sqrt{1 + \frac{1}{4}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)}$$

$$= \frac{a}{4}\left\{e^{\frac{x}{a}} + e^{-\frac{a}{x}}\right\}^2$$

$$= \frac{y^2}{a}$$

*Examples.*

1. Find the equation of the tangent to the parabola $y^2 = 4ax$ at the end of the latus rectum $(x = a)$

*Ans.* $x - y + a = 0$

2. Find the equation of the normal to the parabola $y^2 = 4ax$ at the end of the latus rectum.

*Ans.* $x + y - 3a = 0$

3. Find the length of the subtangent and subnormal to the parabola $y^2 = 4ax$ at the point $(x = a)$

*Ans.* Subtangent $= 2a$

Subnormal $= 2a =$ constant for all points on the curve.

4. Find the equation of the tangent to the curve $y = a^x$ at the point $x = x'$

*Ans.* $y = a^{x'}\{(x - x')\log a + 1\}$

5. Find the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $\left\{\begin{matrix} x = ae \\ y = b\sqrt{1 - e^2}\end{matrix}\right\}$

*Ans.* $bex + a\sqrt{(1 - e^2)}y - ab = 0$

6. Find the length of the subtangent and subnormal to the ellipse mentioned in Example 5, at the point $x = ae$, $y = b\sqrt{1-e^2}$

$$\textit{Ans.}\quad \textit{Subtangent} = \frac{a}{e}(1-e^2)$$

$$\textit{Subnormal} = \frac{eb^2}{a}$$

7. Find the equation of the tangent to the curve $x^a y^b = c$ at the point $x = 1$, $y = \sqrt[b]{c}$

$$\textit{Ans.}\quad y + \frac{a}{b}\sqrt[b]{c}\,x = \sqrt[b]{c}\left(1+\frac{a}{b}\right)$$

8. Find the equation of the tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ at the point $x = a$

$$\textit{Ans.}\quad (c^{\frac{2}{3}} - a^{\frac{2}{3}})^{\frac{1}{2}}x + a^{\frac{1}{3}}y = (c^{\frac{2}{3}} - a^{\frac{2}{3}})^{\frac{1}{2}}c^{\frac{2}{3}}a^{\frac{1}{3}}$$

# CHAPTER X

## RADIUS OF CURVATURE OF PLANE CURVES

LET AB (Fig. 32) be part of a plane curve, and let PO′, QO′ be two normals at the points P and Q intersecting at O′; then if Q approaches indefinitely near to P, O′P and O′Q will be equal to one another, and the circle described with O′ as centre and radius O′P is called the *circle of curvature*, O′P is called the *radius of curvature*, and O′ the centre of curvature of the curve AB at the point P

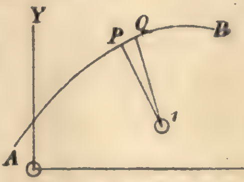


FIG. 32.

Denoting the arc PQ by $ds$ the angle PO′Q by $d\theta$ and O′P by $r$ then

$$ds = r d\theta$$

or

$$\frac{1}{r} = \frac{d\theta}{ds} \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Now, $\theta$ is the angle which the tangent at P makes with the axis of X and $d\theta$ is equal to the angle between the tangents at P and Q We have also

$$\frac{dy}{dx} = \tan\theta$$

hence

$$\frac{d^2y}{dx^2} = \sec^2\theta \frac{d\theta}{dx}$$

since $\theta$ is a function of $x$

therefore $$\frac{d\theta}{dx} = \cos^2\theta\frac{d^2y}{dx^2}$$

Also $$\frac{dx}{ds} = \cos\theta$$

Again, $$\frac{1}{r} = \frac{d\theta}{ds} = \frac{d\theta}{dx}\times\frac{dx}{ds} = \cos^3\theta\frac{d^2y}{dx^2}$$

$$= \frac{\frac{d^2y}{dx^2}}{\sec^3\theta} = \frac{\frac{d^2y}{dx^2}}{\{1+\tan^2\theta\}^{\frac{3}{2}}} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

$$\therefore r = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \ldots\ldots \quad (\beta)$$

Since the numerator of this expression may be positive or negative, it is necessary to give to it the same sign as $\frac{d^2y}{dx^2}$ in order that $r$ may be positive.

If the curve be convex or concave to the axis of X then $\frac{d^2y}{dx^2}$ will be positive or negative respectively, when $y$ is positive.

If $\frac{d^2y}{dx^2} = 0$ the radius of curvature will be infinite. At a point of inflexion where the curve changes from convex to concave to the axis of X or *vice versâ*,

$$\frac{d^2y}{dx^2} = 0$$

It is evident that $\frac{dy}{dx}$ is the same for both the curve and the circle at the point P and $\frac{d^2y}{dx^2}$ is also the same for both,

since the circle and the curve AB have a common normal at the points P and Q

The circle of curvature is said to have contact of the second order, since $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are the same for both.

If two curves given by the equations $y = f(x)$ and $y = \phi(x)$ touch each other at the point $x = a$ then $f'(a) = \phi'(a)$; and if $f''(a) = \phi''(a)$ and also $f'''(a) = \phi'''(a)$ they are said to have contact of the third order; and if $f^n(a) = \phi^n(a)$ they are said to have contact of the $n$th order.

The reciprocal of the radius of curvature is called *curvature.*

In problems relating to the curvature and deflection of beams and girders $\frac{dy}{dx}$ is generally neglected, being a small quantity, and therefore we have

$$\frac{1}{R} = \frac{d^2y}{dx^2}.$$

**To find the radius of curvature of the ellipse whose equation is**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the end of the major axis.

Here $$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{y - x\frac{dy}{dx}}{y^2}$$

Substituting these values in ($\beta$) we get

$$r = \frac{(b^4x^2 + a^4y^2)^{\frac{3}{2}}}{(ab)^4}$$

At the extremity of the major axis

$$x = a \text{ and } y = 0$$

$$\therefore r = \frac{b^2}{a}$$

At the extremity of the minor axis

$$x = 0 \text{ and } y = b$$

$$\therefore r = \frac{a^2}{b}$$

**To find the radius of curvature at the vertex of the parabola** $y^2 = 4ax$

Here $$\frac{dy}{dx} = \frac{2a}{y}$$

and $$\frac{d^2y}{dx^2} = -\frac{2a\frac{dy}{dx}}{y^2} = -\frac{4a^2}{y^3}$$

Substituting in ($\beta$) we have

$$r = \frac{(y^2 + 4a^2)^{\frac{3}{2}}}{4a^2}$$

At the vertex $y = 0$

$$\therefore r = 2a$$

**To find the radii of curvature of the curve**

$$y = (x - 1)(x - 2)(x - 3)$$

at the points where

$$x = 1, x = 2 \text{ and } x = 3$$

Here $$\frac{dy}{dx} = 3x^2 - 12x + 11$$

and $$\frac{d^2y}{dx^2} = 6(x - 2)$$

Where $x = 1$, $\dfrac{dy}{dx} = 2$ and $\dfrac{d^2y}{dx^2} = -6$

$$\therefore r = \frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \frac{(5)^{\frac{3}{2}}}{6}$$

taking the negative root of the numerator.

Where $x = 2$, $\dfrac{d^2y}{dx^2} = 0$

$$\therefore r = \infty$$

a point of inflexion.

Where $x = 3$, $\dfrac{dy}{dx} = 2$ and $\dfrac{d^2y}{dx^2} = 6$

$$\therefore r = \frac{(5)^{\frac{3}{2}}}{6}$$

The radii have the same numerical value at the points $x = 1$ and $x = 3$, but the centres of curvature for the two points are on opposite sides of the curve, since $\dfrac{d^2y}{dx^2}$ is opposite in sign for the two points.

**To find the radius of curvature of the curve** $y = e^x$, where $x = 0$.

Here $\dfrac{dy}{dx} = e^x$, and $\dfrac{d^2y}{dx^2} = e^x$, where $x = 0$, $\dfrac{dy}{dx} = 1$, and $\dfrac{d^2y}{dx^2} = 1$

$$\therefore r = 2^{\frac{3}{2}} = 2\sqrt{2}$$

**To find the radius of curvature of the catenary,**

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$$

at the vertex, $x = 0$

Here $\dfrac{dy}{dx} = \dfrac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}$

and $\dfrac{d^2y}{dx^2} = \dfrac{1}{2a}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\} = \dfrac{y}{a^2}$

$$\therefore\ r = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\frac{y^3}{a^3}}{\frac{y}{a^2}} = \frac{y^2}{a}$$

where $x = 0$, $y = a$; therefore the radius of curvature at the vertex is equal to $a$.

**To find the radius of curvature of the curve** $y = x^4 - 8x^3 + x^2$ at the origin.

Here $\dfrac{dy}{dx} = 4x^3 - 24x^2 + 2x$ and

$$\frac{d^2y}{dx^2} = 12x^2 - 48x + 2$$

At the origin $x = 0$

$$\therefore\ \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} = 2$$

$$\therefore\ r = \tfrac{1}{2}$$

**Find the point of inflexion in the curve**

$$y = x^3 - 15x^2 + 36x + 7$$

We have to find that value of $x$ which will render $\dfrac{d^2y}{dx^2} = 0$

Now, $\dfrac{dy}{dx} = 3x^2 - 30x + 36$

and $\dfrac{d^2y}{dx^2} = 6(x - 5)$

$\therefore\ x = 5$ is the point of inflexion.

**To find the centre of curvature of a curve at any point on it.**

Let the co-ordinates of the centre of the circle, touching the curve AB at P be $(h, k)$

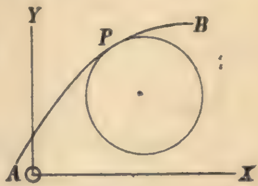

FIG. 33.

Then its equation is

$$(x-h)^2+(y-k)^2 = r^2$$

therefore

$$x-h+(y-k)\frac{dy}{dx} = 0$$

$$1+\left(\frac{dy}{dx}\right)^2+(y-k)\frac{d^2y}{dx^2} = 0$$

therefore
$$k = y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

the ordinate of the centre; and

$$h = x-\frac{\frac{dy}{dx}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}$$

the abscissa of the centre.

**To find the co-ordinates of the centre of curvature of the curve $y = e^{ax}$**

Here $\frac{dy}{dx} = ae^{ax}$ and $\frac{d^2y}{dx^2} = a^2e^{ax}$

$$\therefore\ h = \frac{ax-a^2e^{2ax}-1}{a}$$

and
$$k = \frac{2a^2e^{2ax}+1}{a^2e^{ax}}$$

Where $x = 0$, $h = -\left(a + \frac{1}{a}\right)$

and $k = \frac{2a^2 + 1}{a^2}$

*Examples.*

1. Show that the radius of curvature of a circle is constant, using the general formula.

2. Find the radius of curvature of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at the point $(x'y')$

$$Ans.\quad r = 3(ax'y')^{\frac{1}{3}}$$

3. Find the radius of curvature of the curve whose equation, $y = a \sin x$, at the point $x = \frac{\pi}{2}$

$$Ans.\quad r = \frac{1}{a}$$

4. Find the radius of curvature of the curve whose equation is $y = e^{\tan x}$ at the point $x = \frac{\pi}{4}$

$$Ans.\quad r = \frac{(1 + 4e^2)^{\frac{3}{2}}}{8e}$$

5. Find the radius of curvature of the curve whose equation is $y = x^5 - x^4 - 3x - 1$ at the point $x = 1$

$$Ans.\quad r = \frac{\sqrt{125}}{8}$$

6. Find the radius of curvature of the curve whose equation is

$$y = a \cos^{-1}\left(\frac{a - x}{a}\right) + \sqrt{(2ax - x^2)}$$

at the point $x = a$

$$Ans.\quad r = 2\sqrt{2a}$$

7. Find the co-ordinates of the centre of the curve $y^2 = 4ax$

$$Ans.\quad h = 2a + 3x$$
$$k = -\frac{2x^{\frac{3}{2}}}{\sqrt{a}}$$

8. Find the radius of curvature of the sine curve $y = \sin x$ (1) where $x = \frac{\pi}{2}$ and (2) where $x = \frac{\pi}{4}$

$$Ans.\quad (1)\ R = 1$$
$$(2)\ R = \frac{3\sqrt{3}}{2}$$

CHAPTER XI

# THE CYCLOID, EPICYCLOID AND HYPO-CYCLOID

**The cycloid** *is the path described by a point on the circumference of a circle as it rolls along a straight line.*

Let the circle PD (Fig. 34) roll along the line OX The point P which was initially at O will describe a cycloidal curve OPQ

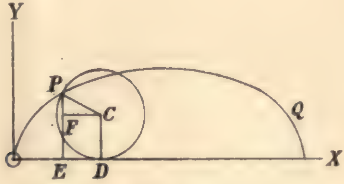


Fig. 34.

Let OE $= x$ and PE $= y$, and the angle PCD $= \theta$

Drop CF perpendicular to PE which is perpendicular to OX Now,

$$y = \text{EF} + \text{FP} = a + a\cos\text{FPC} = a(1 - \cos\theta)$$

where $a$ is the radius of the generating circle. Again,

$$x = \text{OD} - \text{ED} = a\theta - a\sin\theta = a(\theta - \sin\theta)$$

therefore the cycloid is given by the equations

$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}\right\}$$

If we eliminate $\theta$ we have

$$x = a \operatorname{vers}^{-1}\left(\frac{y}{a}\right) - \sqrt{(2ay - y^2)}$$

The direction of the tangent to the cycloid at any point P is $\frac{dy}{dx}$

Now, $dx = a(1 - \cos\theta)d\theta$

and $dy = a \sin\theta \, d\theta$

therefore $\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta} = \cot \frac{1}{2}\theta$

This result suggests a method of drawing a tangent to the cycloid at the point P. Bisect the angle PCD (see Fig. 34) and through P draw a line parallel to the bisector. This line will be a tangent at P to the cycloid.

**The epicycloid** *is the path described by a point on the circumference of a circle as it rolls externally round the circumference of another circle.*

Let ABC be a fixed circle whose centre is taken as origin of co-ordinates, and let its radius be $r$ Let PBG be the rolling circle, and let P be the point on the circumference of the circle PBG which traces out the epicycloid.

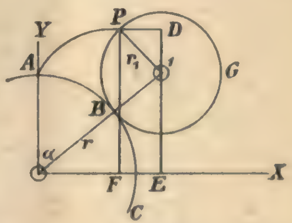

FIG. 35.

The point P at starting coincides with A; therefore the arc AB = arc PB and if the angle AOB = $\alpha$ and the angle PO'B = $\beta$ therefore $r\alpha = r_1\beta$ where $r_1$ is the radius PBG

Let $x$ and $y$ be the co-ordinates of the point P

then $\quad x = \text{OF} = \text{OE} - \text{PD}$

$\therefore \quad x = (r + r_1) \sin a - r_1 \sin (a + \beta)$

and $\quad y = \text{PF} = \text{O'E} + \text{O'D}$

$\therefore \quad y = (r + r_1) \cos a - r_1 \cos (a + \beta)$

If we substitute $\dfrac{ra}{r_1}$ for $\beta$

we get
$$\left.\begin{aligned} x &= (r + r_1) \sin a - r_1 \sin \left(\frac{r + r_1}{r_1}\right)a \\ y &= (r + r_1) \cos a - r_1 \cos \left(\frac{r + r_1}{r_1}\right)a \end{aligned}\right\}$$

If we differentiate, we have

$$\frac{dy}{da} = (r + r_1)\left\{\sin \left(\frac{r + r_1}{r_1}\right)a - \sin a\right\}$$

$$\frac{dx}{da} = (r + r_1)\left\{\cos a - \cos\left(\frac{r + r_1}{r_1}\right)a\right\}$$

$$\therefore \quad \frac{dy}{dx} = \cot\left(\frac{r + 2r_1}{2r_1}\right) a$$

This gives the direction of the tangent to the curve when $a$ is given.

**The hypocycloid** *is the path described by a point on the circumference of a circle as it rolls internally round the circumference of another circle.*

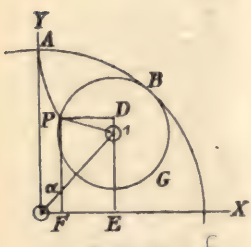


FIG. 36.

Let ABC be the fixed circle, whose centre is taken as origin of co-ordinates, and let its radius be $r$, and let PBG be the rolling circle whose radius is $r_1$ and P the tracing point which coincides with A at starting; therefore the arc

BP $=$ arc AB

Let $a$ denote the angle AOO′ and $\beta$ the angle $\pi - \mathrm{PO'O}$

$$x = \mathrm{OF} = \mathrm{OE} - \mathrm{PD}$$

$$\therefore \quad x = (r - r_1)\sin a - r_1 \sin(a + \pi - \beta)$$

and
$$y = \mathrm{PF} = \mathrm{O'E} + \mathrm{O'D}$$

$$\therefore \quad y = (r - r_1)\cos a - r_1 \cos(a + \pi - \beta)$$

But $ra = r_1\beta$ therefore, substituting $\frac{ra}{r_1}$ for $\beta$ we have

$$\left.\begin{aligned} x &= (r - r_1)\sin a - r_1 \sin\left(\frac{r - r_1}{r_1}\right)a \\ y &= (r - r_1)\cos a + r_1 \cos\left(\frac{r - r_1}{r_1}\right)a \end{aligned}\right\}$$

$$\frac{dy}{da} = -(r - r_1)\left\{\sin a + \sin\left(\frac{r - r_1}{r_1}\right)a\right\}$$

$$\frac{dx}{da} = (r - r_1)\left\{\cos a - \cos\left(\frac{r - r_1}{r_1}\right)a\right\}$$

$$\therefore \quad \frac{dy}{dx} = -\cot\left(\frac{r - 2r_1}{2r_1}\right)a$$

## CHAPTER XII

# THE INTEGRAL CALCULUS

## Integration

THE process of integration may be considered to be the finding of a function when its differential is given.

Suppose we are given $y = x^n$

or $y = x^n + c$ where $c$ is a constant.

On differentiating we obtain

$$\frac{dy}{dx} = nx^{n-1} \text{ in both cases,}$$

therefore $\quad dy = nx^{n-1}dx$

that is, $nx^{n-1}dx$ is the differential of $x^n$ or $x^n + c$ and consequently the integral of $nx^{n-1}dx$ which is denoted by

$$\int nx^{n-1}dx \text{ is } x^n \text{ or } x^n + c$$

where $c$ may be any arbitrary constant. Any function of $x$ when multiplied by $dx$ may be considered a differential, and the object of integration is to obtain a function of $x$ which, on being differentiated, will give that differential.

Simple integration may be considered to be equivalent to finding an area.

Let $y = f(x)$ and let $y$ be plotted vertically and $x$ horizontally, and we get a curve the height being $y$ for a given value of $x$

Now $ydx$ represents the area of the strip the height of

which is $y$ and breadth $dx$, and therefore the integral of $ydx$ or $f(x)dx$, that is, $\int f(x)dx$ represents the area contained by the curve and the axis of $x$, and any two ordinates between which $f(x)$ is integrated. The symbol $\int$ placed immediately before a differential denotes that the expression is to be integrated. It means the sum of all such terms.

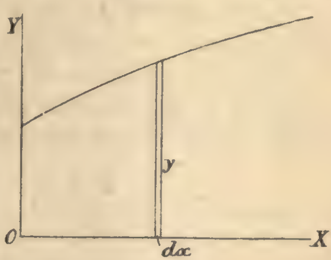


FIG. 37.

The simplest case of integration is that of finding the area of a rectangle the

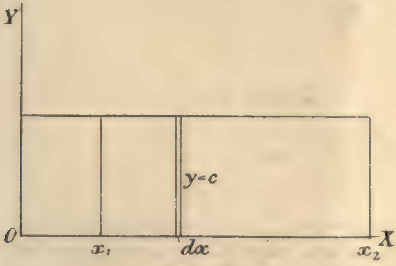


FIG. 38.

height of which is constant where $y = c$ The area is denoted by

$$\int ydx = \int cdx = c\int dx = c[x]$$

since the integral of $dx$ is $x$ The constant is brought outside the sign of integration, since it is common to every term of the elements of area.

If we require the area between any two values, say $x_1$ and $x_2$ we denote the integral thus

$$c\int_{x_1}^{x_2} dx = c\Big[x\Big]_{x_1}^{x_2} = c(x_2 - x_1)$$

$x_2$ being called the superior limit and $x_1$ the inferior limit.

The area is obtained by putting $x_2$ for $x$ in $cx$ and also $x_1$ for $x$ and subtracting as shown above.

Again, suppose $y = x$ and we require to perform the integral $\int ydx = \int xdx$

Now $ydx$ indicates the area of a strip, the height of which is $y$ and breadth $dx$, and therefore $\int ydx$ or $\int xdx$ represents the area of the triangle OAB which is

$$\tfrac{1}{2}\text{OB} \times \text{AB} = \tfrac{1}{2}x_1^2$$

Since OB $= x_1$ and AB $= y_1 = x_1$

$$\therefore \int_0^{x_1} xdx = \left[\frac{x^2}{2}\right]_0^{x_1} = \frac{x_1^2}{2}$$

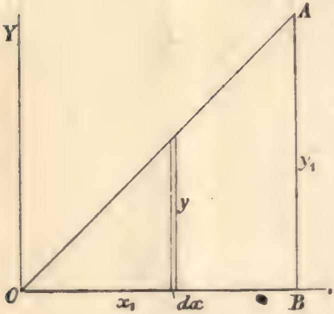


Fig. 39.

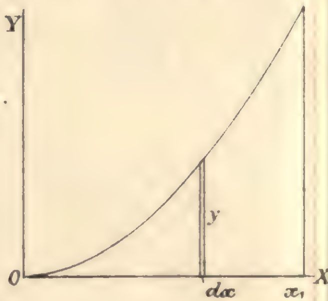


Fig. 40.

Suppose $y = x^2$ and we require the integral of $x^2dx$ between the limits $o$ and $x_1$

that is $$\int_0^{x_1} x^2dx$$

It will be found on differentiating $\frac{x^3}{3}$ that we get $x^2$ and therefore

$$\int_0^{x_1} x^2dx = \left[\frac{x^3}{3}\right]_0^{x_1} = \frac{x_1^3}{3}$$

This can be proved by plotting the curve $y = x^2$ and

measuring the area between the limits where $x = o$ and $x = x_1$

Similarly we find that

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

because on taking the differential of the right-hand side we get

$$x^n dx$$

We therefore deduce the rule for integrating a power of $x$

**Rule.—Increase the index by unity and divide by the index thus increased and add an arbitrary constant.**

From our knowledge of the Differential Calculus we may write down a useful list of Integrals, an arbitrary constant being understood.

1. $\int x^n dx = \dfrac{x^{n+1}}{n+1}$
2. $\int x^{-1} dx = \int \dfrac{dx}{x} = \log x$
3. $\int \dfrac{f'(x)dx}{f(x)} = \log f(x)$
4. $\int e^x dx = e^x$
5. $\int e^{ax} dx = \dfrac{1}{a} e^{ax}$
6. $\int a^x dx = \dfrac{a^x}{\log_e a} = a^x \log_a e$
7. $\int \sin ax dx = -\dfrac{1}{a} \cos ax$
8. $\int \cos ax dx = \dfrac{1}{a} \sin ax$
9. $\int \sec^2 ax dx = \dfrac{1}{a} \tan ax$

√ 10. $\int \operatorname{cosec}^2 ax\,dx = -\frac{1}{a}\cot ax$

11. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log(x+\sqrt{a^2+x^2}) = \sinh^{-1}\frac{x}{a}$

12. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log(x+\sqrt{x^2-a^2}) = \cosh^{-1}\frac{x}{a}$

13. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

14. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$

15. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a}$

16. $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\log\frac{x}{x+\sqrt{x^2+a^2}}$

17. $\int \cosh\frac{x}{a} = a\sinh\frac{x}{a} = a\left(\frac{e^{\frac{x}{a}}-e^{-\frac{x}{a}}}{2}\right)$

18. $\int \sinh\frac{x}{a} = a\cosh\frac{x}{a} = a\left(\frac{e^{\frac{x}{a}}+e^{-\frac{x}{a}}}{2}\right)$

19. $\int \sec x\,dx = \log(\tan x+\sec x) = \int \frac{dx}{\cos x}$

20. $\int \operatorname{cosec} x\,dx = \log\tan\frac{x}{2} = \int \frac{dx}{\sin x}$

21. $\int \tan x\,dx = -\log(\cos x) = \log(\sec x)$

It is recommended that the student should, if possible, commit to memory the list of Integrals given above, although the first three on the list cover a high percentage of most problems that occur in practice, either in the form given or when slightly modified.

When the integral of an expression which does not appear on the list of integrals is required, we should endeavour to simplify it in such a manner that it may become similar to one or more of the forms given on the list.

There are several methods by which this may be done: (1) by substitution or changing the independent variable; (2) by integration by parts; (3) by the method of partial fractions; (4) by the method of successive reduction, etc.

A few examples on these methods will make them clear.

**Substitution Method.**

Suppose we require the integral of the expression

$$\int (ax + b)^n dx$$

This expression does not appear on the list of integrals, but we can reduce it to number (1) on the list by assuming

$$v = ax + b \quad \therefore \frac{dv}{a} = dx$$

and therefore the transformed expression is

$$\frac{1}{a}\int v^n dv = \frac{1}{a}\frac{v^{n+1}}{n+1}$$

$$= \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}$$

*Example.* $$\int \frac{x^2 dx}{(ax - b)^3}$$

Let $v = ax - b \quad \therefore dx = \dfrac{dv}{a}$

and $x = \dfrac{v + b}{a}$

On substituting we have

$$\int\frac{x^2dx}{(ax-b)^3} = \frac{1}{a^3}\int\frac{(v+b)^2dv}{v^3}$$

$$= \frac{1}{a^3}\int\left(\frac{1}{v}+\frac{2b}{v^2}+\frac{b^2}{v^3}\right)dv$$

$$= \frac{1}{a^3}\left[\int\frac{dv}{v}+2b\int v^{-2}dv+b^2\int v^{-3}dv\right]$$

$$= \frac{1}{a^3}\left[\log v - 2bv^{-1} - \frac{b^2v^{-2}}{2}\right]$$

$$= \frac{1}{a^3}\left[\log(ax-b) - \frac{2b}{ax-b} - \frac{b^2}{2(ax-b)^2}\right]$$

*Example.* $\int\frac{dx}{x^2+2bx+c^2}$, $c$ being greater than $b$

By completing the square we have

$$\int\frac{dx}{(x+b)^2+c^2-b^2}$$

Let $v = x+b$, and we obtain

$$\int\frac{dv}{v^2+c^2-b^2} = \frac{1}{\sqrt{c^2-b^2}}\tan^{-1}\frac{v}{\sqrt{c^2-b^2}}$$

$$= \frac{1}{\sqrt{c^2-b^2}}\tan^{-1}\frac{x+b}{\sqrt{c^2-b^2}} \quad \text{by (13) on list.}$$

If $c$ were less than $b$ in this example we should use the method of partial, as there would be linear factors for the denominator.

*Example.* $\int\frac{dx}{\sqrt{x^2+2bx+c^2}}$

On completing the square we have

$$\int\frac{dx}{\sqrt{(x+b)^2+c^2-b^2}}$$

$$= \int\frac{dv}{\sqrt{v^2+c^2-b^2}} \quad \text{where } v = x+b$$

$$= \log\left(v+\sqrt{v^2+c^2-b^2}\right) \quad \text{by (11) on list}$$

$$= \log\left(x+b+\sqrt{x^2+2bx+c^2}\right)$$

*Example.* $\int \frac{dx}{\sqrt{c^2 + 2bx - x^2}} = \int \frac{dx}{\sqrt{c^2 + b^2 - (x-b)^2}}$

$$= \int \frac{dv}{\sqrt{c^2 + b^2 - v^2}}, \text{ where } v = x - b$$

The integral of this expression is

$$\sin^{-1} \frac{v}{\sqrt{c^2 + b^2}} = \sin^{-1} \frac{x - b}{\sqrt{c^2 + b^2}}$$

*Example.* $\int \frac{dx}{x\sqrt{x^2 + 2bx + c}}$

In this or any similar example, let $v = \frac{1}{x}$

$$\therefore dv = -\frac{dx}{x^2} \quad \therefore \frac{dx}{x} = -\frac{dv}{v}$$

and the integral is transformed into

$$-\int \frac{dv}{v\sqrt{\frac{1}{v^2} + \frac{2b}{v} + c}}$$

$$= -\int \frac{dv}{\sqrt{1 + 2bv + cv^2}} = -\sqrt{c}\int \frac{dv}{\sqrt{(cv + b)^2 + (c - b^2)}}$$

$$= -\frac{1}{\sqrt{c}} \log (cv + b + \sqrt{c^2v^2 + 2bcv + c})$$

$$= +\frac{1}{\sqrt{c}} \log \frac{x}{bx + c + \sqrt{c^2 + 2bx + cx^2}}$$

*Example.* $\int \frac{(h + kx)dx}{ax^2 + 2bx + c}$

Here

$$\int \frac{(h + kx)dx}{ax^2 + 2bx + c} = \int \frac{\left(kx + \frac{kb}{a} + h - \frac{kb}{a}\right)}{ax^2 + 2bx + c}dx$$

$$= \frac{k}{a}\int \frac{(ax + b)dx}{ax^2 + 2bx + c} + \frac{ah - kb}{a}\int \frac{dx}{ax^2 + 2bx + c}$$

$$= \frac{k}{2a} \log (ax^2 + 2bx + c) + \frac{ah - kb}{a}$$

$$\times \frac{1}{\sqrt{ac - b^2}} \tan^{-1}\left(\frac{ax + b}{\sqrt{ac - b^2}}\right)$$

if $ac - b^2$ be positive. If $ac - b^2$ be negative, the second expression on the right will be

$$\frac{ah - kb}{a} \times \frac{1}{2\sqrt{b^2 - ac}} \log \frac{ax + b - \sqrt{b^2 - ac}}{ax + b + \sqrt{b^2 - ac}}$$

**To integrate** $\dfrac{dx}{(ax^2 + b)^{\frac{3}{2}}}$

Let $v = \dfrac{1}{x}$

therefore $dx = -\dfrac{dv}{v^2}$ and $ax^2 + b = \dfrac{a + bv^2}{v^2}$

Substituting in terms of $v$ the required integral becomes

$$-\int \frac{vdv}{(bv^2 + a)^{\frac{3}{2}}}$$

Again, assume $bv^2 + a = z^2$ therefore $vdv = \dfrac{zdz}{b}$ and we get

$$-\int \frac{vdv}{(bv^2 + a)^{\frac{3}{2}}} = -\frac{1}{b}\int \frac{zdz}{z^3} = -\frac{1}{b}\int \frac{dz}{z^2}$$

$$= \frac{1}{bz} = \frac{1}{b(bv^2 + a)^{\frac{1}{2}}} = \frac{x}{b(ax^2 + b)^{\frac{1}{2}}}$$

**To integrate** $\dfrac{dx}{\sin x}$

Here $$\int \frac{dx}{\sin x} = \int \frac{\sin x dx}{1 - \cos^2 x}$$

Let $v = \cos x$ therefore $dv = -\sin x dx$ and on substituting in terms of $v$ we have

$$\int \frac{dx}{\sin x} = -\int \frac{dv}{1 - v^2} = \frac{1}{2} \log \frac{1 - v}{1 + v}$$

$$= \log \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \log \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \log \tan \frac{x}{2}$$

**To integrate** $\dfrac{dx}{\cos x}$

Here $\displaystyle\int\frac{dx}{\cos x} = \int\frac{\cos x\,dx}{1-\sin^2 x} = \int\frac{dv}{1-v^2}$

where $v = \sin x$ and

$$\int\frac{dv}{1-v^2} = \frac{1}{2}\log\frac{1+v}{1-v} = \log\sqrt{\frac{1+\sin x}{1-\sin x}}$$

$$= \log\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

**To integrate** $\dfrac{dx}{b+c\cos x}$

$$\int\frac{dx}{b+c\cos x} = \int\frac{dx}{b+c\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$$

$$= \frac{dx}{\int(b+c)\cos^2\frac{x}{2}+(b-c)\sin^2\frac{x}{2}} = \int\frac{\sec^2\frac{x}{2}\,dx}{(b+c)+(b-c)\tan^2\frac{x}{2}}$$

Let $v = \tan\frac{x}{2}$ and the expression becomes

$$2\int\frac{dv}{(b+c)+(b-c)v^2} = \frac{2}{\sqrt{b^2-c^2}}\tan^{-1}\left\{\sqrt{\frac{b-c}{b+c}}\tan\frac{x}{2}\right\}$$

if $b$ be greater than $c$

If $b$ be less than $c$ we have

$$2\int\frac{dv}{(b+c)+(b-c)v^2} = \frac{1}{\sqrt{c^2-b^2}}\log\frac{\sqrt{c+b}-\sqrt{c-b}\tan\frac{x}{2}}{\sqrt{c+b}-\sqrt{c-b}\tan\frac{x}{2}}$$

*Examples on the method of substitution.*

1. $\int x^3 dx$ *Ans.* $\dfrac{x^4}{4}$

2. $\int (x^3 - 3x^2 + 2x + 1)dx$ *Ans.* $\dfrac{x^4}{4} - x^3 + x^2 + x$

3. $\int \left(2x^{\frac{1}{3}} - \dfrac{1}{x} + \sin x\right)dx$ *Ans.* $\dfrac{3}{2}x^{\frac{4}{3}} - \log x - \cos x$

4. $\int (a^x + a^{-x})dx$ *Ans.* $(a^x - a^{-x}) \log_a e$

5. $\int \dfrac{dx}{1 - x}$ *Ans.* $-\log(1 - x)$

6. $\int (e^{3x} - a^{2x} - 3)dx$ *Ans.* $\frac{1}{3}e^{3x} - \frac{1}{2}a^{2x} \log_a e - 3x$

7. $\int \dfrac{xdx}{1 + x^2}$ *Ans.* $\frac{1}{2} \log(1 + x^2)$

8. $\int \dfrac{x^2dx}{1 + x^3}$ *Ans.* $\frac{1}{3} \log(1 + x^3)$

9. $\int \dfrac{\cos xdx}{1 + \sin x}$ *Ans.* $\mathrm{Log}(1 + \sin x)$

10. $\int \dfrac{e^{-x}dx}{a + be^{-x}}$ *Ans.* $-\dfrac{1}{b} \log(a + be^{-x})$

11. $\int \dfrac{xdx}{\sqrt{1 + x^2}}$ *Ans.* $\sqrt{1 + x^2}$

12. $\int \dfrac{x^{-1}dx}{\log x}$ *Ans.* $\mathrm{Log}(\log x)$

13. $\int \sin ax \sin bxdx$

*Ans.* $\dfrac{1}{2} \dfrac{\sin(a - b)x}{a - b} - \dfrac{1}{2} \dfrac{\sin(a + b)x}{a + b}$

14. $\int \dfrac{(1 + x)^3dx}{x^2}$ *Ans.* $\dfrac{x^2}{2} + 3x + 3 \log x - \dfrac{1}{x}$

15. $\int \dfrac{x^4dx}{(1 + x)^2}$

*Ans.* $\dfrac{(x + 1)^3}{3} - 2(x + 1)^2 + 6(x + 1) - 4 \log(x + 1) - \dfrac{1}{x + 1}$

16. $\int \frac{x^7 dx}{\sqrt{a^{16} - x^{16}}}$ *Ans.* $\frac{1}{8} \sin^{-1}\left(\frac{x}{a}\right)^8$

17. $\int \frac{dx}{x^2(1+x)^2}$ Assume $v = \frac{1}{x} + 1$

*Ans.* $2 \log \frac{1+x}{x} - \frac{1+2x}{x(1+x)}$

18. $\int \frac{dx}{x\sqrt{2x-1}}$ *Ans.* $2 \tan^{-1}\sqrt{(2x-1)}$

19. $\int \frac{dx}{x^2 - 16}$ *Ans.* $\frac{1}{8} \log\left(\frac{x-4}{x+4}\right)$

20. $\int \frac{dx}{x^2 + 11x + 30}$ *Ans.* $\text{Log}\left(\frac{x+5}{x+6}\right)$

21. $\int \frac{dx}{x^2 + 2x + 5}$ *Ans.* $\frac{1}{2} \tan^{-1}\left(\frac{x+1}{2}\right)$

22. $\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$

*Ans.* $\text{Log}\,\{x + 1 + \sqrt{(x^2 + 2x + 5)}\}$

23. $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$ *Ans.* $\text{Sin}^{-1}\left(\frac{x+1}{2}\right)$

24. $\int \frac{dx}{x\sqrt{(x^2-1)}}$ *Ans.* $\text{Cos}^{-1}\left(\frac{1}{x}\right)$

25. $\int \frac{dx}{x\sqrt{4 + 4x + 3x^2}}$

*Ans.* $\frac{1}{2} \log\left(\frac{x}{2 + x + \sqrt{4 + 4x + 3x^2}}\right)$

26. $\int \frac{dx}{(2 + 3x^2)^{\frac{3}{2}}}$ *Ans.* $\frac{x}{2(2 + 3x^2)^{\frac{1}{2}}}$

27. $\int \frac{(3 + 4x)dx}{5x^2 + 2x + 2}$

*Ans.* $\frac{2}{5} \log(5x^2 + 2x + 2) + \frac{11}{15} \tan^{-1}\left(\frac{5x+1}{3}\right)$

28. $\int \frac{(1 + e^x + \cos x - \sin x)dx}{x + e^x + \sin x + \cos x}$

*Ans.* $\operatorname{Log}(x + e^x + \sin x + \cos x)$

29. $\int \sqrt{\frac{a - x}{a + x}}dx$ *Ans.* $a \sin^{-1}\left(\frac{x}{a}\right) + \sqrt{(a^2 - x^2)}$

30. $\int \frac{dx}{\sqrt{(x + a)} + \sqrt{(x + b)}}$

*Ans.* $\frac{2}{3(a - b)}\{(x + a)^{\frac{3}{2}} - (x + b)^{\frac{3}{2}}\}$

31. $\int \sin^2 (ax)dx$ *Ans.* $\frac{x}{2} - \frac{\sin 2ax}{4a}$

32. $\int \cos^2 (ax)ax$ *Ans.* $\frac{x}{2} + \frac{\sin 2ax}{4a}$

33. $\int \sin^7 x dx$ *Ans.* $\frac{\cos^7 x}{7} - \frac{3 \cos^5 x}{5} + \cos^3 x - \cos x$

34. $\int \cos^5 (ax + b)dx$

*Ans.* $\frac{1}{a}\left\{\frac{\sin^5 (ax + b)}{5} - \frac{2}{3} \sin^3 (ax + b) + \sin (ax + b)\right\}$

35. Given $V = V_0 \sin pt$ and $C = C_0 \sin pt$ find the average value of VC

*Ans.* $\frac{V_0C_0}{2}$

36. Given $V = V_0 \sin (pt + \theta)$ and $C = C_0 \sin pt$ find the average value of VC

*Ans.* $\frac{V_0C_0}{2} \cos \theta$

37. Find the average value of

$(a + b \sin pt + c \sin 2pt + d \sin 3pt \ldots)^2$

*Ans.* $a^2 + \frac{b^2}{2} + \frac{c^2}{2} + \frac{d^2}{2} \ldots$

## Method of Integration by Parts.

The integral of any expression of the form

$$\int udv$$

can be made to depend on another integral of the form $\int vdu$

Let $y = uv$ where $u$ and $v$ are functions of $x$

then $\qquad dy = udv + vdu = d(uv)$

and therefore $\qquad \int udv + \int vdu = uv$

On transposing we have the general formula for integrating by parts

$$\int \mathbf{udv} = \mathbf{uv} - \int \mathbf{vdu}$$

This method has to be adopted in a great many examples where the integral of a product of two functions is required, such as $\int x^n e^x dx \quad \int x^2 \sin xdx \quad \int e^{ax} \sin bxdx$

A few examples will illustrate the method.

*Example.* $\qquad \int x \cos xdx$

In this example let $u = x$ and therefore $dv = \cos xdx$

$$\therefore du = dx \text{ and } v = \int \cos xdx = \sin x$$

$$\therefore \int x \cos xdx = x \sin x - \int \sin xdx$$
$$= x \sin x + \cos x$$

In the above example, suppose $u = \cos x$ and therefore $dv = xdx \quad \therefore v = \dfrac{x^2}{2}$ and $du = -\sin xdx$

and the integral would be

$$\int x \cos xdx = \frac{x^2}{2} \cos x + \frac{1}{2}\int x^2 \sin xdx$$

In this case the integral on the right hand is more complicated than the original, which shows that the assumption was a wrong one. In general assume $u$ to be that part of the expression which becomes simpler, where possible, on being differentiated.

*Example.* $\int e^{at} \sin pt dt$

$$\text{Let } u = e^{at} \text{ and therefore } \sin pt dt = dv$$

$$du = ae^{at}dt \text{ and } v = -\frac{1}{p}\cos pt$$

$$\therefore \int e^{at} \sin pt dt = -\frac{1}{p}e^{at}\cos pt + \frac{a}{p}\int e^{at}\cos pt dt \quad . \quad (a)$$

Similarly $$\int e^{at}\cos pt dt = \frac{1}{p}e^{at}\sin pt - \frac{a}{p}\int e^{at}\sin pt dt$$

Substituting in $(a)$ we have

$$\int e^{at}\sin pt dt = -\frac{1}{p}e^{at}\cos pt + \frac{a}{p}\left[\frac{1}{p}e^{at}\sin pt - \frac{a}{p}\int e^{at}\sin pt dt\right]$$

On transposing the integral term on the right we have

$$\left(1+\frac{a^2}{p^2}\right)\int e^{at}\sin pt dt = \frac{e^{at}}{p^2}\{a\sin pt - p\cos pt\}$$

$$\therefore \int e^{at}\sin pt dt = \frac{e^{at}}{a^2+p^2}\{a\sin pt - p\cos pt\}$$

$$= \frac{e^{at}}{\sqrt{a^2+p^2}}\sin(pt-\theta)$$

where $$\tan\theta = \frac{p}{a}$$

This is an important example, as it has several practical applications.

There is a much easier method of arriving at the same result by the method of operators. *See examples on operators.*

*Example.* $\sqrt{(r^2-x^2)}dx$

Here $$\int\sqrt{r^2-x^2}dx = \int\frac{(r^2-x^2)dx}{\sqrt{r^2-x^2}}$$

$$= r^2\int\frac{dx}{\sqrt{r^2-x^2}} - \int\frac{x^2dx}{\sqrt{r^2-x^2}}$$

Therefore

$$\int\sqrt{r^2-x^2}dx = r^2\sin^{-1}\left(\frac{x}{r}\right)-\int\frac{x^2dx}{\sqrt{r^2-x^2}} \quad . \quad (a)$$

Again, on integrating by parts we get

$$\int\sqrt{(r^2-x^2)}dx = x\sqrt{r^2-x^2}+\int\frac{x^2dx}{\sqrt{r^2-x^2}} \quad . \quad (b)$$

On adding ($a$) and ($b$) and dividing by 2 we get

$$\int\sqrt{(r^2-x^2)}dx = \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right)+\frac{x\sqrt{r^2-x^2}}{2}$$

Similarly, we may show that

$$\int\sqrt{(x^2-r^2)}dx = -\frac{r^2}{2}\log\{x+\sqrt{(x^2-r^2)}\}+\frac{x}{2}\sqrt{(x^2-r^2)}$$

These two integrals may be used for finding the areas of a circle and hyperbola whose equations are $x^2+y^2 = r^2$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2} = 1$ respectively.

**To integrate** $x^2e^xdx$

Let $x^2 = u$ and $e^xdx = dv$ therefore $v = e^x$ and $du = 2xdx$ on substituting in the formula for integrating by parts, we have

$$\int x^2e^xdx = x^2e^x - 2\int xe^xdx$$

Again,

$$\int xe^xdx = xe^x - \int e^xdx = xe^x - e^x$$

therefore

$$\int x^2e^xdx = x^2e^x - 2xe^x + 2e^x = e^x(x^2-2x+2)$$

**To integrate** $x\tan^{-1}xdx$

Let $\frac{x^2}{2} = u$ and $\tan^{-1}x = v$ therefore $dv = \frac{dx}{1+x^2}$

and on substituting in the formula we have

$$\int x\tan^{-1}xdx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2dx}{1+x^2}$$

Again,

$$\int \frac{x^2 dx}{1+x^2} = \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \tan^{-1} x$$

therefore

$$\int x \tan^{-1} x dx = \frac{x^2}{2} \tan^{-1} x + \frac{\tan^{-1} x}{2} - \frac{x}{2}$$

*Examples.*

1. $\int \log ax dx$ *Ans.* $x \log ax - x$

2. $\int x^m \log x dx$ *Ans.* $\frac{x^{m+1}}{m+1}\left\{\log x - \frac{1}{m+1}\right\}$

3. $\int \theta \tan^2 \theta d\theta$ *Ans.* $\theta \tan \theta + \log(\cos \theta) - \frac{\theta^2}{2}$.

4. $\int \sqrt{\left(\frac{a+x}{x}\right)} dx$

*Ans.* $\sqrt{ax + x^2} + a \log\{\sqrt{x} + \sqrt{(a+x)}\}$.

5. $\int (1 + \cos x)^2 dx$ *Ans.* $\frac{3x}{2} + 2 \sin x + \frac{\sin 2x}{4}$

6. $\int e^x \sin px \cos qx dx$

*Ans.* $$\frac{e^x}{2}\left\{\frac{\sin(p+q)x - (p+q)\cos(p+q)x}{1+(p+q)^2} + \frac{\sin(p-q)x - (p-q)\cos(p-q)x}{1+(p-q)^2}\right\}$$

7. $\int \frac{\theta + \sin\theta}{1 + \cos\theta} d\theta$ *Ans.* $\theta \tan \frac{\theta}{2}$

8. $\int \frac{\log(\log\theta) d\theta}{\theta}$ *Ans.* $\text{Log}\, \theta \times \log(\log\theta) - \log\theta$

9. $\int \sqrt{2x - x^2} dx$

*Ans.* $\frac{x-1}{2}\sqrt{(2x - x^2)} + \frac{1}{2}\sin^{-1}(x-1)$

10. $\int \frac{dx}{1+e^x}$ *Ans.* $\text{Log} \frac{e^x}{e^x + 1}$

11. $\int x^n(\log x)^2 dx$

*Ans.* $\dfrac{x^{n+1}}{n+1}\left\{\dfrac{2}{(1+n)^2} - \dfrac{2}{1+n}\log x + (\log x)^2\right\}$

12. $\displaystyle\int \frac{\log x dx}{(1-x)^2}$ *Ans.* $\dfrac{x \log x}{1-x} + \log(1-x)$

## Integration by Partial Fractions.

The integration of a rational fraction, such as

$$\int \frac{(3x+4)dx}{(x+1)(x+2)}$$

where the denominator consists of two or more rational factors of the first degree, is obtained by resolving the expression into its partial fractions (see p. 2), thus

$$\begin{aligned}\int \frac{(3x+4)dx}{(x+1)(x+2)} &= \int\left(\frac{1}{x+1} + \frac{2}{x+2}\right)dx \\ &= \log(x+1) + 2\log(x+2)\end{aligned}$$

*Example.* $\displaystyle\int \frac{(4x^2 - 7x + 6)dx}{(x+1)(x-2)(x-3)}$

Let $\dfrac{f(x)}{\phi(x)} = \dfrac{4x^2 - 7x + 6}{(x+1)(x-2)(x-3)} = \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

On clearing of fractions we have

$$\begin{aligned}4x^2 - 7x + 6 &= A(x-2)(x-3) + B(x+1)(x-3) \\ &\quad + C(x+1)(x-2)\end{aligned}$$

Both sides of this equation are identical for all values of $x$ therefore if we substitute 3 for $x$ we get

$$f(3) = C(3+1)(3-2) = C\phi'(3)$$

$$\therefore C = \frac{f(3)}{\phi'(3)} = \frac{21}{4}$$

Similarly $\qquad B = \dfrac{f(2)}{\phi'(2)} = \dfrac{8}{-3}$

and $$A = \frac{f(-1)}{\phi'(-1)} = \frac{17}{12}$$

$$\therefore \int \frac{(4x^2 - 7x + 6)dx}{(x+1)(x-2)(x-3)} = \frac{17}{12}\int \frac{dx}{x+1} - \frac{8}{3}\int \frac{dx}{x-2} + \frac{21}{4}\int \frac{dx}{x-3}$$

$$= \frac{17}{12}\log(x+1) - \frac{8}{3}\log(x-2) + \frac{21}{4}\log(x-3)$$

*Example.* $$\int \frac{x^5 dx}{x^3 - 7x - 6}$$

By division we get

$$\frac{x^5}{x^3 - 7x - 6} = x^2 + 7 + \frac{6x^2 + 49x + 42}{(x+1)(x+2)(x-3)}$$

and

$$\frac{f(x)}{\phi(x)} = \frac{6x^2 + 49x + 42}{(x+1)(x+2)(x-3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x-3}$$

also $\phi'(x) = 3x^2 - 7$

Therefore $$A = \frac{f(-1)}{\phi'(-1)} = \frac{1}{4}$$

$$B = \frac{f(-2)}{\phi'(-2)} = -\frac{32}{5}$$

and $$C = \frac{f(3)}{\phi'(3)} = \frac{243}{20}$$

therefore

$$\int \frac{x^5 dx}{x^3 - 7x - 6} = \int x^2 dx + \int 7dx + \frac{1}{4}\int \frac{dx}{x+1}$$
$$- \frac{32}{5}\int \frac{dx}{x+2} + \frac{243}{20}\int \frac{dx}{x-3}$$

$$= \frac{x^3}{3} + 7x + \frac{1}{4}\log(x+1) - \frac{32}{5}\log(x+2) + \frac{243}{20}\log(x-3)$$

*Example.* $$\int \frac{x^2 dx}{(x-1)^2(x+2)(x-3)}$$

Assume $$\frac{x^2}{(x-1)^2(x+2)(x-3)}$$

$$= \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x+2} + \frac{D}{x-3}$$

and by the method shown above it will be found that

$A = -\frac{1}{6}$, $B = -\frac{13}{36}$, $C = -\frac{4}{45}$ and $D = \frac{9}{20}$

therefore $$\int \frac{x^2dx}{(x-1)^2(x+2)(x-3)}$$

$$= -\frac{1}{6}\int\frac{dx}{(x-1)^2} - \frac{13}{16}\int\frac{dx}{x-1} - \frac{4}{45}\int\frac{dx}{x+2} + \frac{9}{20}\int\frac{dx}{x-3}$$

$$= \frac{1}{6(x-1)} - \frac{13}{16}\log(x-1) - \frac{4}{45}\log(x+2) + \frac{9}{20}\log(x-3)$$

*Example.* $$\int \frac{x^2dx}{(x^2+2)(x-1)^2}$$

Assume

$$\frac{x^2}{(x^2+2)(x-1)^2} = \frac{Ax+B}{x^2+2} + \frac{C}{(x-1)^2} + \frac{D}{x-1}$$

Clearing of fractions, and equating coefficients of like powers of $x$

we get $A = -\frac{4}{9}$, $B = \frac{2}{9}$, $C = \frac{1}{3}$ and $D = \frac{4}{9}$

therefore $$\int \frac{x^2dx}{(x^2+2)(x-1)^2}$$

$$= -\frac{2}{9}\int\frac{(2x-1)dx}{x^2+2} + \frac{1}{3}\int\frac{dx}{(x-1)^2} + \frac{4}{9}\int\frac{dx}{x-1}$$

$$= -\frac{2}{9}\log(x^2+2) + \frac{\sqrt{2}}{9}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{1}{3(x-1)} + \frac{4}{9}\log(x-1)$$

**Integrate** $$\frac{dx}{(x^2+x+1)(x^2+1)}$$

Assume

$$\frac{1}{(x^2+x+1)(x^2+1)} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2+1}$$

Clearing of fractions, and equating the coefficients of like powers of $x$

we get $A = 1$, $B = 1$, $C = -1$ and $D = 0$

therefore

$$\int\frac{dx}{(x^2+x+1)(x^2+1)} = \int\frac{(x+1)dx}{x^2+x+1} - \int\frac{xdx}{x^2+1}$$

$$= \frac{1}{2}\log(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) - \frac{1}{2}\log(x^2+1)$$

$$= \frac{1}{2}\log\left(\frac{x^2+x+1}{x^2+1}\right) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

**Integrate** $\dfrac{xdx}{x^4+x^2-2}$

Assume $\dfrac{x}{x^4+x^2-2} = \dfrac{Ax+B}{x^2+2} + \dfrac{C}{x+1} + \dfrac{D}{x-1}$

Clearing of fractions, and equating coefficients of like powers of $x$

we get $A = -\frac{1}{3}$, $B = 0$, $C = \frac{1}{6}$ and $D = \frac{1}{6}$

therefore

$$\int\frac{xdx}{x^4+x^2-2} = -\frac{1}{3}\int\frac{xdx}{x^2+2} + \frac{1}{6}\int\frac{dx}{x+1} + \frac{1}{6}\int\frac{dx}{x-1}$$

$$= -\frac{1}{6}\log(x^2+2) + \frac{1}{6}\log(x+1) + \frac{1}{6}\log(x-1)$$

$$= \frac{1}{6}\log\left(\frac{x^2-1}{x^2+2}\right)$$

This example admits of an easier solution by assuming $x^2 = z$

*Examples.*

1. $\int \frac{dx}{1-x^3}$

*Ans.*

$$\frac{1}{6}\log(1+x+x^2)-\frac{1}{3}\log(1-x)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

2. $\int \frac{(x-1)dx}{x^2-4}$ *Ans.* $\log(x+2)^{\frac{3}{4}}(x-2)^{\frac{1}{4}}$

3. $\int \frac{x^3dx}{x^2+6x+8}$

*Ans.* $\frac{x^2}{2}-6x+32\log(x+4)-4\log(x+2)$

4. $\int \frac{(x^2-2x+3)dx}{x^2-3x+2}$

*Ans.* $x+3\log(x-2)-2\log(x-1)$

5. $\int \frac{xdx}{x^3+x^2+x+1}$

*Ans.* $\frac{1}{2}\tan^{-1}x+\frac{1}{4}\log(1+x^2)-\frac{1}{2}\log(1+x)$

6. $\int \frac{(x^2+1)dx}{x^4+x^2+1}$

*Ans.* $\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$

7. $\int \frac{dx}{(1+x)(1+x^2)(1+2x)}$

*Ans.*

$$\tfrac{4}{5}\log(2x+1)-\tfrac{3}{20}\log(x^2+1)-\tfrac{1}{10}\tan^{-1}x-\tfrac{1}{2}\log(x+1)$$

8. $\int \frac{xdx}{x^2-x-2}$ *Ans.* $\frac{2}{3}\log(x-2)+\frac{1}{3}\log(x+1)$

9. $\int \frac{x^2dx}{x^4+3x^2+2}$ *Ans.* $\sqrt{2}\tan^{-1}\frac{x}{\sqrt{2}}-\tan^{-1}x$

10. $\int \frac{(3x-4)dx}{(x+2)(x+1)^2}$ *Ans.* $10\log\left(\frac{x+1}{x+2}\right)+\frac{7}{x+1}$

11. $\displaystyle\int\frac{x^2dx}{x^4+4x^3+5x^2+4x+4}$

*Ans.*

$$\frac{2}{25}\log(1+x^2)-\frac{4}{25}\log(x+2)-\frac{4}{5(x+2)}-\frac{3}{25}\tan^{-1}x$$

12. $\displaystyle\int\frac{dx}{(1-x^3)^{\frac{1}{3}}}$ Assume $1-x^3 = y^3x^3$

*Ans.* $\dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{2y-1}{\sqrt{3}}\right)-\log(1+y)$

where $y^3 = \dfrac{1-x^3}{x^3}$

13. $\displaystyle\int\frac{x^3dx}{x^4+1}$ *Ans.* $\dfrac{1}{4}\log(x^4+1)$

14. $\displaystyle\int\frac{(1+x^2)dx}{x(1+x^2+x^4)}$

*Ans.* $\displaystyle \text{Log}\, x-\frac{1}{4}\log(x^2-x+1)+\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

$$-\frac{1}{4}\log(x^2+x+1)-\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

15. $\displaystyle\int\frac{dx}{(x^2+p^2)(x+q)}$

*Ans.* $\displaystyle\frac{1}{p^2+q^2}\left\{\log(x+q)-\frac{1}{2}\log(x^2+p^2)+\frac{q}{p}\tan^{-1}\left(\frac{x}{p}\right)\right\}$

16. $\displaystyle\int\frac{\theta^3d\theta}{\theta^6+1}$

*Ans.*

$$\frac{1}{2\sqrt{3}}\{\tan^{-1}(2\theta-\sqrt{3})-\tan^{-1}(2\theta+\sqrt{3})\}+\frac{1}{12}\log\frac{1-\theta^2+\theta^4}{(1+\theta^2)^2}$$

### Integration by Successive Reduction.

Suppose we require the integral of $\sin^n x dx$ where $n$ is a positive integer. If $n$ be odd, the required integral is easily obtained; for let $z = \cos x$ therefore $dz = -\sin x dx$

and $\sin^n xdx = \sin^{n-1} x \sin xdx = -\sin^{n-1} xd(\cos x)$ and substituting in terms of $z$ we get $\sin^n xdx = -(1-z^2)^{\frac{n-1}{2}} dz$ but $\frac{n-1}{2}$ is an even integer if $n$ be greater than unity, and after expanding by the Binomial Theorem the integral is easily obtained.

Similarly, if the integral of $\cos^n xdx$ is required where $n$ is an odd positive integer and greater than unity, let $z = \sin x$ $\therefore dz = \cos xdx$ also $\cos^n xdx = \cos^{n-1} x \cos xdx = \cos^{n-1} xd(\sin x)$ therefore by substitution

$$\cos^n xdx = (1-z^2)^{\frac{n-1}{2}} dz$$

which is easily integrable since $\frac{n-1}{2}$ is an even positive integer.

**Integrate** $\sin^7 xdx$

Here, $\sin^7 xdx = \sin^6 x \sin xdx = -\sin^6 xd(\cos x) = -(1-z^2)^3dz$ where $z = \cos x$ therefore

$$\int\sin^7 xdx = -\int(1-z^2)^3dz = -\int(1-3z^2+3z^4-z^6)dz$$

$$= -z+z^3-\frac{3}{5}z^5+\frac{z^7}{7}$$

$$= -\cos x+\cos^3 x-\frac{3}{5}\cos^5 x+\frac{1}{7}\cos^7 x$$

**Integrate** $\cos^5 xdx$

Here,

$$\cos^5 xdx = \cos^4 x \cos xdx = (1-\sin^2 x)^2d(\sin x)$$
$$= (1-z^2)^2dz \text{ where } z = \sin x \text{ therefore}$$

$$\int\cos^5 xdx = \sin x-\tfrac{2}{3}\sin^3 x+\tfrac{1}{5}\sin^5 x$$

Suppose we require the integral of $\sin^n xdx$ where $n$ is an even positive integer.

We have

$$\int\sin^n xdx = \int\sin^{n-1} x \sin xdx = -\int\sin^{n-1} xd(\cos x)$$

Integrating by parts we have

$$\int \sin^n x dx = -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x dx$$
$$= -\cos x \sin^{n-1} x + (n-1)\int (1-\sin^2 x) \sin^{n-2} x dx$$

and by transposing and dividing by $n$ we have

$$\int \sin^n x dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x dx$$

By successive application of this result to the integral on the right-hand side, we get the required result.

**Integrate** $\sin^2 x dx$

$$\int \sin^2 x dx = \int \frac{(1-\cos 2x)dx}{2} = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x dx$$

therefore $$\int \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4}$$

**Integrate** $\cos^2 x dx$

$$\int \cos^2 x dx = \int \frac{(1+\cos 2x)dx}{2} = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x dx$$

therefore $$\int \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4}$$

**Integrate** $\sin^4 x dx$

We have

$$\int \sin^4 x dx = -\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x dx$$
$$= -\frac{\cos x \sin^3 x}{4} + \frac{3x}{8} - \frac{3}{16}\sin 2x$$

**Integrate** $\cos^4 x dx$

We have

$$\int \cos^4 x dx = \frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x dx$$
$$= \frac{\sin x \cos^3 x}{4} + \frac{3x}{8} + \frac{3}{16}\sin 2x$$

Suppose we require the integral of $\sin^n x \cos^m x dx$ where $n$ and $m$ are both positive integers.

First let $m$ be odd. The expression may be written thus:

$$\int \sin^n x \cos^{m-1} x \cos x dx = \int \sin^n x \cos^{m-1} x d(\sin x)$$

Substitute $z$ for $\sin x$ and we have

$$\int \sin^n x \cos^m x dx = \int z^n (1 - z^2)^{\frac{m-1}{2}} dz$$

and since $\frac{m-1}{2}$ is an even integer, the expression is easily integrated after expanding by the binomial theorem.

Again, suppose $n$ and $m$ both positive and even.

$$\text{Let } v = \frac{\sin^{n+1} x}{n+1} \text{ and } u = \cos^{m-1} x$$

therefore $dv = \sin^n x \cos x dx$ and integrating by parts we have

$$\int \sin^n x \cos^m x dx = \frac{\sin^{n+1} x \cos^{m-1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{n+2} x \cos^{m-2} x dx$$

By repeated application of this formula the required integral can be obtained.

**To integrate** $e^{ax} x^n dx$

Integrating by parts we have

$$\int e^{ax} x^n dx = \frac{1}{a} e^{ax} x^n - \frac{n}{a} \int e^{ax} x^{n-1} dx$$

$$= \frac{1}{a} e^{ax} x^n - \frac{n}{a^2} e^{ax} x^{n-1} + \frac{n(n-1)}{a^2} \int e^{ax} x^{n-2} dx \text{ etc.}$$

**To integrate $e^{ax} \sin^n x dx$**

Integrating by parts we have

$$\int e^{ax} \sin^n x dx = \frac{1}{a} e^{ax} \sin^n x - \frac{n}{a} \int e^{ax} \sin^{n-1} x \cos x dx$$

$$= \frac{1}{a} e^{ax} \sin^n x - \frac{n}{a^2} e^{ax} \sin^{n-1} x \cos x$$

$$+ \frac{n}{a^2} \int e^{ax} \{(n-1) \sin^{n-2} x (1 - \sin^2 x) - \sin^n x\} dx$$

Transposing and dividing by $a^2 + n^2$ we have

$$\int e^{ax} \sin^n x dx = \frac{e^{ax}}{a^2 + n^2} \sin^{n-1} x (a \sin x - n \cos x)$$

$$+ \frac{n(n-1)}{a^2 + n^2} \int e^{ax} \sin^{n-2} x dx$$

Repeating this process on the integral on the right-hand side we, finally, obtain the required integral.

**To integrate** $\dfrac{dx}{\sin^n x}$ and $\dfrac{dx}{\cos^n x}$

$$\int \frac{dx}{\sin^n x} = -\frac{\cos x}{(n-1) \sin^{n-1} x} + \frac{n-2}{n-1} \int \frac{dx}{\sin^{n-2} x}$$

$$\int \frac{dx}{\sin^{10} x} = -\frac{\cos x}{9 \sin^9 x} + \frac{8}{9} \int \frac{dx}{\sin^8 x} \text{ etc.}$$

Similarly,

$$\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1) \cos^{n-1} x} + \frac{(n-2)}{(n-1)} \int \frac{dx}{\cos^{n-2} x}$$

$$\int \frac{dx}{\cos^8 x} = \frac{\sin x}{7 \cos^7 x} + \frac{6}{7} \int \frac{dx}{\cos^6 x} \text{ etc.}$$

*Examples.*

1. $\int \sin^3 x dx$ *Ans.* $\dfrac{\cos^3 x}{3} - \cos x$

2. $\int \sin^5 x dx$ *Ans.* $-\dfrac{\cos^5 x}{5} + \dfrac{2}{3} \cos^3 x - \cos x$

3. $\int \cos^3 x dx$ *Ans.* $-\frac{\sin^3 x}{3} + \sin x$

4. $\int \cos^5 x dx$ *Ans.* $\frac{\sin^5 x}{5} - \frac{2}{3}\sin^3 x + \sin x$

5. $\int \sin^6 x dx$

*Ans.* $-\frac{\cos x \sin^5 x}{6} - \frac{5}{24}\cos x \sin^3 x - \frac{5}{16}\cos x \sin x + \frac{5x}{16}$

6. $\int \cos^6 x dx$

*Ans.* $\frac{\text{Sin } x \cos^5 x}{6} - \frac{5 \sin x \cos^3 x}{24} + \frac{5 \sin x \cos x}{16} + \frac{5x}{16}$

7. $\int \sin x \cos^3 x dx$ *Ans.* $\frac{\text{Sin}^2 x}{2} - \frac{\sin^4 x}{4}$

8. $\int \sin^2 x \cos^5 x dx$ *Ans.* $\frac{\text{Sin}^3 x}{3} - \frac{2}{5}\sin^5 x + \frac{\sin^7 x}{7}$

9. $\int \sin^4 x \cos^7 x dx$

*Ans.* $\frac{\text{Sin}^5 x}{5} - \frac{3}{7}\sin^7 x + \frac{1}{3}\sin^9 x - \frac{1}{11}\sin^{11} x$

10. $\int \sin x \cos^2 x dx$ *Ans.* $-\frac{\cos^3 x}{3}$

11. $\int \sin^3 x \cos^4 x dx$ *Ans.* $\frac{\text{Cos}^7 x}{7} - \frac{\cos^5 x}{5}$

12. $\int \sin^5 x \cos^6 x dx$

*Ans.* $-\frac{\cos^7 x}{7} + \frac{2}{9}\cos^9 x - \frac{1}{11}\cos^{11} x$

13. $\int \sin^2 x \cos^2 x dx$ *Ans.* $\frac{x}{8} - \frac{\sin 4x}{32}$

14. $\int \sin^4 x \cos^2 x dx$

*Ans.* $\frac{\text{Sin}^5 x \cos x}{6} - \frac{\sin^3 x \cos x}{24} - \frac{\sin 2x}{32} + \frac{x}{16}$

15. $\int \sin^2 x \cos^4 x dx$

*Ans.* $-\dfrac{\sin x \cos^5 x}{6} + \dfrac{1}{24} \sin x \cos^3 x + \dfrac{1}{16} \sin x \cos x + \dfrac{x}{16}$

16. $\int \sin^4 x \cos^4 x dx$

*Ans.* $-\dfrac{\sin^3 x \cos^5 x}{8} - \dfrac{1}{16} \sin x \cos^5 x + \dfrac{\sin x \cos^3 x}{64}$

$$+ \frac{3}{128} \sin x \cos x + \frac{3x}{128}$$

17. $\int \tan^6 x dx$ *Ans.* $\dfrac{\tan^5 x}{5} - \dfrac{\tan^3 x}{3} + \tan x - x$

Here, $\int \tan^6 x dx = \int \tan^4 x(\sec^2 x - 1)dx$
$= \int \tan^4 x d(\tan x) - \int \tan^4 x dx$ etc.

18. $\int e^{ax} x^4 dx$

*Ans.* $\dfrac{e^{ax}}{a}\left(x^4 - \dfrac{4x^3}{a} + \dfrac{4 \times 3x^2}{a^2} - \dfrac{4 \times 3 \times 2x}{a^3} + \dfrac{\lfloor 4}{a^4}\right)$

19. $\int e^{3x} \sin^3 x dx$

*Ans.* $\dfrac{e^{3x}}{6}\left\{\sin^2 x(\sin x - \cos x) + \dfrac{1}{5}(3 \sin x - \cos x)\right\}$

20. $\displaystyle\int \frac{dx}{\sin^5 x}$ *Ans.* $-\dfrac{\cos x}{4 \sin^4 x} - \dfrac{3}{8}\dfrac{\cos x}{\sin^2 x} + \dfrac{3}{8} \log \tan \dfrac{x}{2}$

21. $\displaystyle\int \frac{dx}{\cos^3 x}$ *Ans.* $\dfrac{\sin x}{2 \cos^2 x} + \dfrac{1}{4} \log\left(\dfrac{1 + \sin x}{1 - \sin x}\right)$

## CHAPTER XIII

# DEFINITE INTEGRALS

We shall now endeavour to give an elementary idea of a definite integral, regarding the process of integration as a summation.

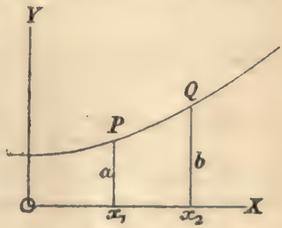


FIG. 41.

Suppose $y$ the ordinate of the curve PQ (Fig. 41) to vary continuously by successive increments, being $a$ at the point P and $b$ at the point Q. It is evident that the total increment of $y$ is $b - a$ and considering this total increment as the sum of its partial increments, we have

$$\int_a^b dy = b - a$$

$b$ being called the superior limit and $a$ the inferior limit of $y$

Again, suppose $y$ a function of $x$

the relation being $\quad y = \phi(x)$

and let $x = x_1$ where $y = a$

and $x = x_2$ where $y = b$

therefore $a = \phi(x_1)$ and $b = \phi(x_2)$

Also $\quad dy = \phi'(x)dx$

therefore $\quad \int_{x_1}^{x_2} \phi'(x)dx = \phi(x_2) - \phi(x_1)$

$x_2$ is the superior limit and $x_1$ the inferior limit of $x$

We see that, in order to integrate a continuous function of $x$ between certain limits we first find the indefinite integral, and then substitute for $x$ the superior limit and also the inferior limit in the indefinite integral, and subtract the latter from the former.

**Find the value of** $\int x^3dx$ between the limits $x = 2$ and $x = 4$

$$\int_2^4 x^3dx = \left[\frac{x^4}{4}\right]_2^4 = \frac{4^4}{4} - \frac{2^4}{4} = 64 - 4$$

therefore $$\int_2^4 x^3dx = 60$$

**Find the value of the definite integral** $\int_0^4 \frac{xdx}{1+x^2}$

The integral of $\frac{xdx}{1+x^2}$ is $\frac{1}{2}\log(1+x^2)$

Substitute 4 for $x$ and also 0 for $x$ and subtract the latter from the former; therefore

$$\int_0^4 \frac{xdx}{1+x^2} = \frac{1}{2}\{\log 17 - \log 1\} = \frac{1}{2}\log 17$$

The complete discussion of definite integrals is beyond the scope of this elementary treatise.

*Examples.*

1. $\int_0^2 x^n dx$ *Ans.* $\frac{2^{n+1}}{n+1}$
2. $\int_1^2 \log x dx$ *Ans.* $2\log_e 2 - 1$
3. $\int_0^\pi \sin x dx$ *Ans.* $2$
4. $\int_0^{\frac{\pi}{2}} \cos x dx$ *Ans.* $1$
5. $\int_0^\pi \sin^2 x dx$ *Ans.* $\frac{\pi}{2}$

6. $\int_0^{\frac{\pi}{2}} \cos^2 x dx$ *Ans.* $\frac{\pi}{4}$

7. $\int_0^1 \frac{dx}{1+x^2}$ *Ans.* $\frac{\pi}{4}$

8. $\int_0^1 (x^3 - 3x^2 + 2x + 1)dx$ *Ans.* $\frac{5}{4}$

9. $\int_2^3 \frac{xdx}{1+x^2}$ *Ans.* $\operatorname{Log}\sqrt{2}$

10. $\int_0^{\frac{\pi}{2}} \frac{\cos x dx}{1+\sin x}$ *Ans.* $\operatorname{Log} 2$

11. $\int_0^a \frac{x^7 dx}{\sqrt{a^{16} - x^{16}}}$ *Ans.* $\frac{\pi}{16}$

12. $\int_{\frac{1}{2}}^1 \frac{dx}{x\sqrt{2x-1}}$ *Ans.* $\frac{\pi}{2}$

13. $\int_6^{16} \frac{dx}{x^2 - 16}$ *Ans.* $\frac{1}{8}\log 3$

14. $\int_0^a \sqrt{\frac{a-x}{a+x}}\,dx$ *Ans.* $\frac{\pi a}{2} - a$

15. $\int_0^r \sqrt{r^2 - x^2}\,dx$ *Ans.* $\frac{\pi r^2}{4}$

## CHAPTER XIV

# AREAS OF PLANE CURVES

SUPPOSE we require the area bounded by the curve PQ the axis of X and the ordinates P$a$ and Q$b$ the axes being rectangular. Let $y$ be the ordinate at the point C and let $y + dy$ be the ordinate at the point D indefinitely near to C let $x$ be the abscissa of C and $x + dx$ that of D

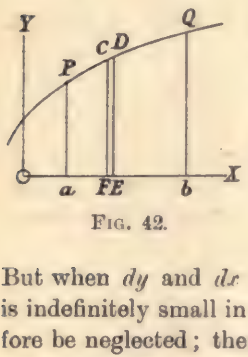


FIG. 42.

Then the area CDEF $= \frac{1}{2}(y + y + dy)dx = d$A where A is the area required, that is

$$dA = ydx + \tfrac{1}{2}dydx$$

But when $dy$ and $dx$ are indefinitely small, the term $\frac{1}{2}dydx$ is indefinitely small in comparison with $ydx$ and may therefore be neglected; therefore

$$dA = ydx$$

Integrating we get

$$A = \int ydx + C$$

where C is a constant.

Again, suppose $y$ and $x$ connected by the equation $y = \phi(x)$

Therefore $$A = \int\phi(x)dx + C$$

But we want the area between the limits $x = a$ and $x = b$

therefore $$A = \int_a^b \phi(x)dx = \Big[f(x)\Big]_a^b$$

where $f(x)$ is the integral of $\phi(x)dx$

Therefore $$A = f(b) - f(a)$$

The constant C disappears, because when integrating between limits the constant C is always subtracted from itself.

The general formula for the area bounded by a continuous curve and the axis of X is

$$A = \int y dx + C \quad . \quad . \quad . \quad . \quad (A)$$

The general formula for the area bounded by any continuous curve and the axis of Y is

$$A = \int x dy + C \quad . \quad . \quad . \quad . \quad (B)$$

The axes in both cases being rectangular.

**Find the area bounded by the curve whose equation is $y = x^2$ and the axis of X between the limits $x = 1$ and $x = 4$**

By formula (A) we have

$$A = \int y dx + C = \int_1^4 x^2 dx + C$$

that is, $$A = \left[\frac{x^3}{3} + C\right]_1^4$$

$$\therefore A = \left(\frac{4^3}{3} + C\right) - \left(\frac{1^3}{3} + C\right) = \frac{64}{3} - \frac{1}{3}$$

$$\therefore A = 21$$

**Find the area bounded by the curve mentioned in the last example and the axis of Y between the same limits, viz. $x = 1$ and $x = 4$**

By formula (B) we have

$$A = \int x\,dy + C\text{, but } dy = 2x\,dx$$

$$\therefore A = \int 2x^2 dx + C$$

that is
$$A = \left[\frac{2x^3}{3} + C\right]_1^4$$

$$\therefore A = \left(\frac{2 \times 4^3}{3} + C\right) - \left(\frac{2 \times 1^3}{3} + C\right)$$

$$\therefore A = 42$$

**Find the area of the parabola $y^2 = 4ax$ between the limits $x = 0$ and $x = a$**

By (A) we have

$$A = \int y\,dx + C = 2\sqrt{a}\int_0^a \sqrt{x}\,dx + C$$

that is
$$A = 2\sqrt{a} \times \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a = \left[\frac{4\sqrt{ax} \times x}{3}\right]_0^a = \frac{2}{3}xy$$

That is, the area bounded by the parabola $y^2 = 4ax$ the axis of X and the ordinate at the point where $x = a$ is

$\frac{2}{3}$ (area of the circumscribing rectangle)

The area bounded by the curve and the axis of Y between the same limits is

$\frac{1}{3}$ (area of the circumscribing rectangle)

**Area of a circle.**

Let the area of the circle be divided into concentric rings the breadth of each measured radially being $dx$, and let the radius of an element be $x$ then the length of an element being $2\pi x$ and breadth $dx$ its area is $2\pi x\,dx$

The whole area of the circle

$$= \int_0^r 2\pi x\,dx = 2\pi\left[\frac{x^2}{2}\right]_0^r = \pi r^2$$

*Another method.*—Suppose the area to be divided into an indefinite number of sectors the angle of each being $d\theta$, then an element of area is

$$\frac{r^2}{2}d\theta$$

$$\therefore \text{ whole area } = \frac{r^2}{2}\int_0^{2\pi} d\theta = \frac{r^2}{2}\Big[\theta\Big]_0^{2\pi} = \pi r^2$$

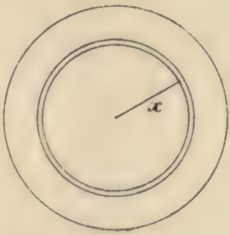


FIG. 43.


FIG. 44.

The area may also be found as follows:—

The equation to the circle is $x^2 + y^2 = r^2$ when the centre is at the origin, and an element of area is expressed by

$$y\,dx = \sqrt{r^2 - x^2}\,dx$$

$$\therefore \text{ area of a quadrant } = \int_0^r \sqrt{r^2 - x^2}\,dx$$

$$= \frac{1}{2}\left\{x\sqrt{r^2 - x^2} + r^2 \sin^{-1}\frac{x}{r}\right\}_0^r$$

$$= \frac{\pi r^2}{4}$$

therefore the whole area $= \pi r^2$

**Area of the ellipse whose equation is** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Here $$y = \frac{b}{a}\sqrt{(a^2 - x^2)}$$

and $\frac{1}{4}$ the required area is

$$A = \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}dx$$

that is $$A = \frac{b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

that is $$A = \frac{b}{a} \times \frac{\pi a^2}{4} = \frac{\pi ab}{4}$$

Therefore the whole area of the ellipse is $4A = \pi ab$

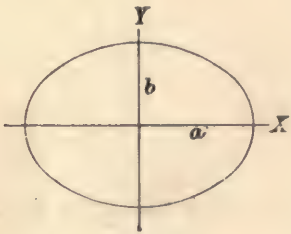


FIG. 45.

**Area of the cycloid.**

$$\left.\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}\right\} \text{(see page 122)}$$

The area $A = \int y dx + C$ that is,

$$A = a^2\int_0^{2\pi}(1 - \cos\theta)^2 d\theta = a^2\int(1 - 2\cos\theta + \cos^2\theta)d\theta$$

$$= a^2\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$\therefore A = 3\pi a^2$$

Therefore the area of a cycloid is three times the area of the generating circle.

## The Catenary

The catenary is the curve assumed by an inelastic string or chain of uniform density suspended by its extremities.

We shall investigate the equation of the catenary.

Let AOPB (Fig. 46) be the catenary.

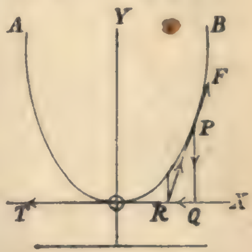

FIG. 46.

Taking the vertex as origin, OY and OX as the axis of Y and X respectively.

Consider the equilibrium of the arc OP

There are three forces acting on it, viz. the horizontal tension T at the vertex, the tangential force F at P and the weight of OP which we shall denote by S assuming unit length to be of unit weight. These three forces meet at a point, and are proportional to the three sides of the triangle PQR

that is $$F : T : S :: PR : RQ : QP$$

Also $$F = \sqrt{T^2 + S^2}$$

Now $$\frac{dy}{dS} = \frac{S}{F} = \frac{S}{\sqrt{T^2 + S^2}}$$

$$\therefore y = \int \frac{S\,dS}{\sqrt{T^2 + S^2}} + C$$

where C is a constant.

$$\therefore y = \sqrt{T^2 + S^2} + C$$

But $y = 0$ where $S = 0$ therefore $C = -T$

Therefore $$y + T = \sqrt{T^2 + S^2} \quad . \quad . \quad . \quad \text{(A)}$$

We have also $$\frac{dx}{dS} = \frac{T}{\sqrt{T^2 + S^2}}$$

Therefore $$x = \mathrm{T}\int \frac{d\mathrm{S}}{\sqrt{\mathrm{T}^2 + \mathrm{S}^2}} + \mathrm{C}$$

that is $$x = \mathrm{T} \log \{\mathrm{S} + \sqrt{\mathrm{T}^2 + \mathrm{S}^2}\} + \mathrm{C}$$

and $\mathrm{S} = 0$ where $x = 0$ therefore $\mathrm{C} = -\mathrm{T} \log \mathrm{T}$

therefore $$x = \mathrm{T} \log \left\{\frac{\mathrm{S} + \sqrt{\mathrm{T}^2 + \mathrm{S}^2}}{\mathrm{T}}\right\}$$

therefore $$\mathrm{S} + \sqrt{\mathrm{T}^2 + \mathrm{S}^2} = \mathrm{T}e^{\frac{x}{\mathrm{T}}} \quad . \quad . \quad . \quad (\mathrm{B})$$

Again $$\{\sqrt{\mathrm{T}^2 + \mathrm{S}^2} - \mathrm{S}\}\{\sqrt{\mathrm{T}^2 + \mathrm{S}^2} + \mathrm{S}\} = \mathrm{T}^2$$

$$\therefore \sqrt{\mathrm{T}^2 + \mathrm{S}^2} - \mathrm{S} = \frac{\mathrm{T}^2}{\sqrt{\mathrm{T}^2 + \mathrm{S}^2} + \mathrm{S}} = \frac{\mathrm{T}^2}{\mathrm{T}e^{\frac{x}{\mathrm{T}}}} = \mathrm{T}e^{-\frac{x}{\mathrm{T}}} \quad (\mathrm{C})$$

Adding (B) and (C) we have

$$2\sqrt{\mathrm{T}^2 + \mathrm{S}^2} = \mathrm{T}\left\{e^{\frac{x}{\mathrm{T}}} + e^{-\frac{x}{\mathrm{T}}}\right\}$$

therefore $$\sqrt{\mathrm{T}^2 + \mathrm{S}^2} = \frac{\mathrm{T}}{2}\left\{e^{\frac{x}{\mathrm{T}}} + e^{-\frac{x}{\mathrm{T}}}\right\}$$

therefore by (A) we get

$$y + \mathrm{T} = \frac{\mathrm{T}}{2}\left\{e^{\frac{x}{\mathrm{T}}} + e^{-\frac{x}{\mathrm{T}}}\right\}$$

If we move the axis of X down a distance equal to the length of string or chain whose weight is equal to T the equation becomes

$$y = \frac{\mathrm{T}}{2}\left\{e^{\frac{x}{\mathrm{T}}} + e^{-\frac{x}{\mathrm{T}}}\right\}$$

which may be written in the form

$$y = \mathrm{T} \cosh \frac{x}{\mathrm{T}}$$

**Area bounded by the catenary curve and the axis of $x$ between the limits $x = 0$, and $x = x_1$**

$$\mathrm{A} = \int y\,dx = \frac{\mathrm{T}}{2}\int_0^{x_1}\left(e^{\frac{x}{\mathrm{T}}} + e^{-\frac{x}{\mathrm{T}}}\right)dx$$

$$\therefore \mathrm{A} = \frac{\mathrm{T}^2}{2}\left[e^{\frac{x}{\mathrm{T}}} - e^{-\frac{x}{\mathrm{T}}}\right]_0^{x_1}$$

$$\therefore \mathrm{A} = \frac{\mathrm{T}^2}{2}\left(e^{\frac{x_1}{\mathrm{T}}} - e^{-\frac{x_1}{\mathrm{T}}}\right) = \mathrm{T}^2 \sinh \frac{x}{\mathrm{T}}$$

**Areas in Terms of Polar Co-ordinates.**

Let $x$ and $y$ be the rectangular co-ordinates of the point P on the curve AB Then, if we take OX as the initial line, O being the pole, the angle POX $= \theta$ the vectorial angle, and OP the radius vector $\rho$ of P we have $x = \rho\cos\theta$ and $y = \rho\sin\theta$

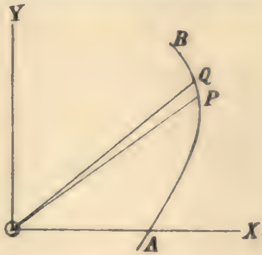

FIG. 47.

Let the point Q be indefinitely near to P and denote the angle POQ by $d\theta$ then an element of the area of the curve is $\frac{1}{2}$OP . OQ sin POQ but when P and Q are indefinitely near OP $=$ OQ and denoting the element of the area by $d$A we have

$$dA = \tfrac{1}{2}\rho^2 d\theta$$

since $\sin d\theta = d\theta$ when $d\theta$ is indefinitely small; therefore the area of any continuous curve in polar co-ordinates is given by the formula

$$A = \tfrac{1}{2}\int \rho^2 d\theta$$

when taken between the proper limits.

**Area of the Curve whose Equation is** $\rho = a(1 + \cos\theta)$ (*Cardioid*). It is obvious that the initial line divides it symmetrically, and that the whole area is

$$A = \int \rho^2 d\theta$$

that is, $$A = a^2\int_0^\pi (1 + \cos\theta)^2 d\theta$$

therefore $$A = a^2\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^\pi$$

$$\therefore A = \frac{3\pi a^2}{2}$$

**Area of the Lemniscate whose Equation is** $\rho^2$

$= a^2 \cos 2\theta$ This curve (Fig. 48) has two loops, and its whole area is therefore

$$A = 4 \times \tfrac{1}{2}\int \rho^2 d\theta$$

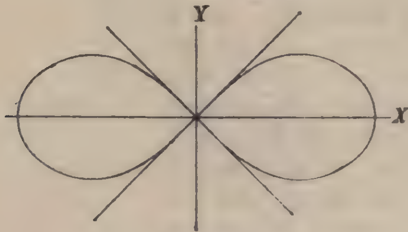


FIG. 48.

$$\therefore A = 2a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta d\theta = 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

that is, $A = 2a^2 \times \tfrac{1}{2} = a^2$

*Examples.*

1. Find the area of the curve whose equation is $\rho = a \cos \theta$ *Ans.* $\dfrac{\pi a^2}{4}$

2. Find the area of a loop of the curve $\rho^2 = a^2 \cos 6\theta$ This curve has 6 loops. *Ans.* $\dfrac{a^2}{6}$

3. Find the area bounded by the curve whose equation is $y = ae^{kx}$ and the axis of X between the limits $x = 0$ and $x = b$ *Ans.* $\dfrac{a}{k}(e^{kb} - 1)$

4. Find the area contained by the curve $y = x^3 - 3x^2 + 4x$ and the axis of X *Ans.* $\dfrac{x^4}{4} - x^3 + 2x^2$

5. Find the area included between the curves $y^2 = 4mx$ $x^2 = 4my$ *Ans.* $\dfrac{16m^2}{3}$

6. Find the whole area of the ellipse whose equation is $ax^2 + 2hxy + by^2 = 1$ *Ans.* $\dfrac{\pi}{\sqrt{ab - h^2}}$

7. Find the area of the curve $m^2y^4 = x^4(m^2 - x^2)$ between the limits $x = m$ and $x = -m$

*Ans.* $\dfrac{8m^2}{5}$

8. Find the whole area between the curve $xy^2 = 4m^2 \times (2m - x)$ and the line $y = 0$ *Ans.* $4\pi m^2$

## CHAPTER XV

# SURFACES OF REVOLUTION

If a plane curve revolve about a fixed straight line in its plane, every point in the curve will, obviously, describe a circle, and the curve will generate a surface called a *surface of revolution.*

The line about which the curve revolves is called the *axis* of revolution.

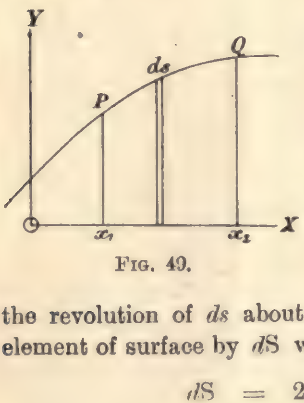

Fig. 49.

The section of a surface of revolution, by a plane perpendicular to its axis, is a circle.

Let the curve PQ (Fig. 49) revolve about the axis of X then we may assume that the arc PQ is made up of an infinite number of straight lines of length $ds$

The surface generated by the revolution of $ds$ about OX is $2\pi y ds$ and denoting this element of surface by $dS$ we have

$$dS = 2\pi y ds$$

therefore
$$S = 2\pi \int y ds + C$$

Again,
$$ds = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx$$

therefore
$$S = 2\pi \int y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} dx + C \quad . \quad (A)$$

If $y = f(x)$ be the equation of the curve PQ (Fig. 49) then the surface generated by PQ is

$$S = 2\pi\int_{x_1}^{x_2} f(x)\sqrt{\left\{1 + [f'(x)]^2\right\}}\,dx$$

Similarly, we may show that the surface generated by the revolution of a plane curve about the axis of Y is

$$S = 2\pi\int x\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx + C \quad . \quad (B)$$

**Surface of a sphere.**

The surface of a sphere is generated by the revolution of a circle about a diameter.

Let the circle, whose centre is at the origin, revolve about the axis of X then its equation is $x^2 + y^2 = r^2$

$$\therefore y = \sqrt{r^2 - x^2}$$

By formula (A) we have

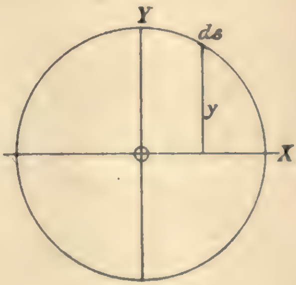


FIG. 50.

$$S = 2\pi\int_{-r}^{+r}\sqrt{r^2 - x^2}\sqrt{\left\{1 + \frac{x^2}{r^2 - x^2}\right\}}\,dx$$

that is $\quad S = 2\pi\int_{-r}^{+r} r\,dx = 2\pi\Big[rx\Big]_{-r}^{+r}$

$$\therefore S = 2\pi\{r^2 - (-r^2)\} = 4\pi r^2$$

It will be noticed that the surface of a sphere is equal to the curved surface of its circumscribing cylinder, and the surface of a zone on a sphere is equal to the curved surface of a cylinder of the same radius as the sphere and height equal to that of the zone.

**Surface generated by the revolution of the parabola $y^2 = 4ax$ about the axis of X between the limits $x = 0$ and $x = a$**

$$S = 2\pi\int\sqrt{4ax}\sqrt{\left\{1+\frac{a}{x}\right\}}dx + C \text{ by A}$$

Therefore $$S = 4\pi\sqrt{a}\int_0^a\sqrt{(x+a)}dx$$

that is, $$S = 8\pi\sqrt{a}\left[\frac{(x+a)^{\frac{3}{2}}}{3}\right]_0^a$$

$$\therefore S = \frac{8\pi a^2}{3}(2^{\frac{3}{2}} - 1)$$

**Surface generated by the revolution of the catenary $y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}$ about the axis of Y**

By B, we have

$$S = 2\pi\int_0^x x\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}dx$$

Now, $$\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}} = \frac{1}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}$$

therefore $$S = \pi\int_0^x x\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}dx$$

$$\int_0^x x\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}dx = cx\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) - c^2\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + 2c^2$$

$$\therefore S = \pi c\left\{x\left(e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right) - c\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right) + 2c\right\}$$

**Surface generated by the revolution of the curve $y = ae^{cx}$ round the axis of X**

$$S = 2\pi a\int e^{cx}\sqrt{\{1+(ac)^2e^{2cx}\}}dx + C$$

Let $ace^{cx} = \tan\theta$ and the expression becomes

$$S = \frac{2\pi}{c^2}\int\frac{d\theta}{\cos^3\theta} = \frac{2\pi}{c^2}\left\{\frac{\sin\theta}{2\cos^2\theta} + \frac{1}{4}\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right\} + C$$

**Surface generated by the revolution of a right-angled triangle about the side *a***

The side $c$ will generate the surface of a cone whose convex area is required. Denote the semi-vertical angle by $\alpha$ therefore the required area

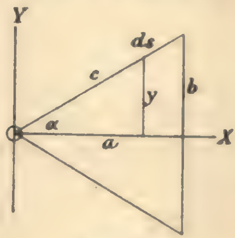

FIG. 51.

$$S = 2\pi\int y\,ds$$

$$= 2\pi\tan\alpha\int_0^a x\sqrt{1+\tan^2\alpha}\,dx$$

since $y = x\tan\alpha$

$$\therefore S = 2\pi\tan\alpha\sqrt{1+\tan^2\alpha}\int_0^a x\,dx$$

that is, $$S = 2\pi\tan\alpha\sqrt{1+\tan^2\alpha}\times\frac{a^2}{2}$$

and since $\tan\alpha = \dfrac{b}{a}$ we have

$$S = \pi b\sqrt{a^2+b^2}$$

that is, the area is the product of semi-circumference of base and slant side.

**Surface of a spherical ring.**

A spherical ring is generated by the revolution of a circle about a straight line in its plane. Let the radius of the ring be $r$, and let the distance of the centre of the circle from the straight line be R Let $aba_1b_1$ (Fig. 52) be the generating circle, and let $aa_1$D be drawn perpendicular to the axis of X then an element of the area of the ring is equal to the sum of

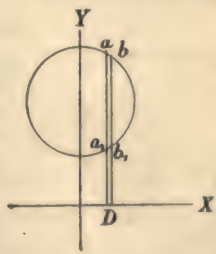

FIG. 52.

the elementary zones generated by the arcs $ab$ and $a_1b_1$

therefore $$dS = 2\pi(y_1 + y_2)ds$$

where $$y_1 = aD \text{ and } y_2 = a_1D$$

that is, $$dS = 4\pi R ds$$

since $$R = \tfrac{1}{2}(y_1 + y_2)$$

because the circle is divided symmetrically by the straight line through its centre parallel to the axis of X

therefore $$S = 4\pi R \int_0^{\frac{s}{2}} ds$$

that is, $$S = 2\pi R s$$

where $s$ is the circumference of the generating circle; that is,

$$S = 2\pi R \times 2\pi r = 4\pi^2 R r$$

In a ring R is always greater than $r$

## Guldinus' Theorems.

*First.* **The surface generated by the revolution of a curve about any external axis in its plane, is equal to the product of the perimeter of the curve into the distance through which the centre of gravity of the curve moves.**

Let $\bar{y}$ denote the distance of the centre of gravity of the perimeter of the curve from the axis, $s$ the length of the curve; then we have, by the method of moments,

$$\bar{y}s = \int y ds$$

therefore $$2\pi\bar{y}s = 2\pi\int y ds$$

This proves the theorem, since $2\pi\bar{y}$ is the distance through which the centre of gravity of the curve moves in one revolution, and $s$ is the length of the curve.

*Second.* **The volume generated by the revolution**

of a plane area about an external axis in its plane is equal to the product of the area into the distance through which its centre of area moves.

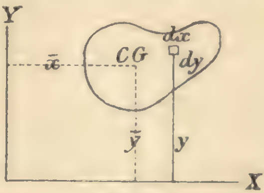

FIG. 53.

To prove this theorem; let the whole area be denoted by A and let OX denote the external axis about which it revolves, and $\bar{x}$ $\bar{y}$ the co-ordinates of its centre of area; then an element of area $dA = dydx$ and we have the moment of the whole area A about OX equal to the sum of the moments of its elements about OX

therefore $\bar{y}A = \int y dA$

that is, $\bar{y}A = \int\int y dx dy = \frac{1}{2}\int y^2 dx$

therefore $2\pi\bar{y}A = \pi\int y^2 dx$

This integral must be taken within the proper limits.

Now $2\pi\bar{y}$ is the distance through which the centre of gravity of the area moves in one revolution, and A is its whole area, which establishes the theorem, since $\pi\int y^2 dx$ represents the whole volume.

**Volume of a ring.**

The volume of a ring is equal to the area of the section of the ring multiplied by the distance through which the centre of area of the section moves in one revolution.

$$\therefore \text{Volume} = 2\pi R \times \pi r^2 = 2\pi^2 R r^2$$

where R is the radius of the ring measured from its centre to the centre of the section.

*Example.*—Find the volume formed by the revolution of a triangle about an axis through its vertex parallel to its base. Let $h$ be the height and $b$ the base, then the area is

$$\tfrac{1}{2}bh$$

and the centre of area is $\frac{2}{3}h$ from the vertex, therefore the required volume is

$$\begin{aligned} & 2\pi \times \tfrac{2}{3}h \times \tfrac{1}{2}bh \\ = & \tfrac{2}{3}\pi bh^2 \end{aligned}$$

*Examples.*

1. Find the surface generated by the revolution of the line $y = mx + c$ about the axis of X between the limits $x = 0$ and $x = a$

*Ans.* $2\pi\sqrt{(1+m^2)}\left(\frac{ma^2}{2} + ac\right)$

2. Find the surface generated by the revolution of the curve $y = ax^3$ about the axis of X

*Ans.* $\frac{\pi}{27a}\left(1 + 9a^2x^4\right)^{\frac{3}{2}} + C$

3. Find the surface generated by the revolution of the cycloid,

$$\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}$$

round its base.

*Ans.* $\frac{64\pi a^2}{3}$

4. Find the surface generated by the revolution of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

about its minor axis.

*Ans.* $\pi\left\{2a^2 + \frac{ab^2}{\sqrt{a^2 - b^2}}\log\left(\frac{a + \sqrt{a^2 - b^2}}{a - \sqrt{a^2 - b^2}}\right)\right\}$

5. Find the surface generated by the revolution of an arc of a circle of radius $r$ subtending an angle of 60° at the

centre, about a line in its plane parallel to its chord and at a distance $d$ from it.

$$\textit{Ans.}\quad \frac{2\pi^2 r}{3}\left\{d + r\sqrt{3}\left(\frac{\sqrt{3}}{\pi} - \frac{1}{2}\right)\right\}$$

6. Find the surface generated by the revolution of a semicircular arc about a tangent at its middle point.

$$\textit{Ans.}\quad 2\pi r^2(\pi - 2)$$

## CHAPTER XVI

# VOLUMES OF SOLIDS OF REVOLUTION

LET a plane curve revolve about the axis of X then the volume enclosed by the surface generated is called a volume of revolution. Let P be any point on the curve at a distance $y$ from the axis of X Then P describes a circle and the area of this circle is $\pi y^2$ An element of volume is the area of this circle multiplied by $dx$ and therefore the volume enclosed by the surface of revolution and by two planes perpendicular to the axis of X through the points where $x = x_1$ and $x = x_2$ is given by

$$V = \pi\int_{x_1}^{x_2} y^2 dx$$

where $y$ is expressed in terms of $x$

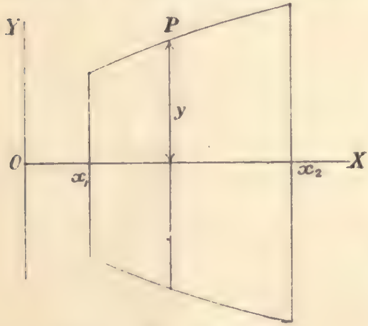


FIG. 54.

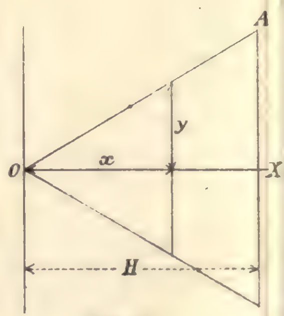


FIG. 55.

**Volume of a cone.**

Let the vertex of the cone be taken as the origin and let

OX be taken as the axis of X The volume is generated by the revolution of the triangle AOX about the axis of X.

Similar △'s $\frac{x}{y} = \frac{H}{R}$ $y = \frac{R}{H}x$

$$\text{Now } y = \frac{R}{H}x$$

$$\therefore \text{Vol.} = \pi\frac{R^2}{H^2}\int_0^H x^2dx = \pi\frac{R^2}{H^2}\left[\frac{x^3}{3}\right]_0^H = \frac{\pi R^2 H}{3}$$

**Volume of a sphere.**

Taking the centre of the sphere as origin, the equation to the generating circle is

$$x^2 + y^2 = R^2$$

$$\therefore y^2 = R^2 - x^2$$

$$\therefore \text{Vol.} = \pi\int_{-R}^{+R}(R^2 - x^2)dx = \pi\left[R^2x - \frac{x^3}{3}\right]_{-R}^{+R}$$

$$= \frac{4\pi R^3}{3}$$

*Another method.*—Imagine the sphere to be divided up into an indefinite number of concentric shells of thickness $dx$ The surface of a shell of radius $x$ is

$$4\pi x^2$$

therefore an element of volume is

$$4\pi x^2dx$$

$$\therefore \text{Vol.} = 4\pi\int_0^R x^2dx = 4\pi\left[\frac{x^3}{3}\right]_0^R = 4\pi\frac{R^3}{3}$$

The volume of a hollow sphere is given by

$$\text{Vol.} = 4\pi\int_r^R x^2dx = 4\pi\left(\frac{R^3 - r^3}{3}\right)$$

**Volume generated by the revolution of the parabola $y^2 = 4ax$ about its axis.**

$$\text{Here} \qquad V = 4\pi\int_0^x axdx$$

therefore $$V = 4\pi a\left[\frac{x^2}{2}\right]_0^x$$

$$= \frac{x}{2} \times 4\pi a x = \pi y^2 \times \frac{x}{2}$$

that is, the volume is equal to one-half the volume of the circumscribing cylinder.

**Volume generated by the revolution of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, about the axis of X**

Here $$dV = \pi a^3(1 - \cos\theta)^3 d\theta$$

therefore $$V = \pi a^3 \int_0^{2\pi} (1 - \cos\theta)^3 d\theta$$

that is

$$V = \pi a^3 \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)d\theta$$

$$= \pi a^3 \left[\theta - 3\sin\theta + \frac{3\theta}{2} + \frac{3\sin 2\theta}{4} - \sin\theta + \frac{\sin^3\theta}{3}\right]_0^{2\pi}$$

$$\therefore\ V = 5\pi^2 a^3$$

*Examples.*

1. Find the volume generated by the revolution of a triangle about a line through its vertex parallel to its base.

*Ans.* $\frac{2}{3}\pi h^2 b$ where $h$ is its height and $b$ its base.

2. Find the volume generated by the revolution of a segment of a circle about its chord.

*Ans.* $2\pi r\left\{\frac{(2r^2 + d^2)\cos\theta}{3} - rd\right\}$ where $r$ is the radius, $d$ the distance of the chord from the centre, and $\sin\theta = \frac{d}{r}$

3. Find the volume generated by the revolution of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the axis of Y

*Ans.* $6\pi^3 a^3$

4. Find the volume of the part cut off from a sphere, of radius $r$ by a plane whose distance is $\frac{r}{3}$ from its centre.

*Ans.* $\frac{28}{81}\pi r^3$

5. Find the volume of the ellipsoid whose equation is $\frac{x^2}{l^2}+\frac{y^2}{m^2}+\frac{z^2}{n^2} = 1$

*Ans.* $\frac{4\pi lmn}{3}$

6. Find the volume of the frustum of a sphere.

*Ans.* $\frac{\pi k}{6}\{k^2+3(a_1{}^2+a_2{}^2)\}$ where $k$ is its height, and $a_1$ and $a_2$ the radii of its ends.

7. Find the ratio of a hemisphere, paraboloid and cone, on the same base and all of the same altitude.

*Ans.* 4 : 3 : 2

8. Find the volume generated by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ about a line in its plane at a distance $d$ from its centre.

*Ans.* $2\pi^2abd$

## CHAPTER XVII

# LENGTHS OF CURVES

THE process of determining the length of an arc of a curve is called its *rectification.*

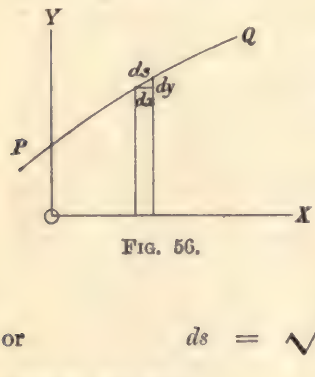


FIG. 56.

Let PQ (Fig. 56) be an arc of a curve whose length is required; then denoting an indefinitely small portion of it by $ds$ we have

$$(ds)^2 = (dx)^2 + (dy)^2$$

therefore

$$ds = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}dx$$

or

$$ds = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}dy$$

therefore the arc

$$s = \int\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}dx = \int\sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}dy \quad \text{(A)}$$

**The length of the circumference of a circle.** Let the equation of the circle be

$$x^2 + y^2 = r^2$$

By (A), we have

$$s = \int\sqrt{\left(1 + \frac{x^2}{y^2}\right)}dx$$

since $$\frac{dy}{dx} = -\frac{x}{y}$$

$$\therefore s = \int \frac{rdx}{\sqrt{r^2 - x^2}}$$

and the whole circumference is equal to four times this integral between the limits $x = 0$ and $x = r$ therefore the whole circumference

$$s = 4\int_0^r \frac{rdx}{\sqrt{r^2 - x^2}}$$

$$= 4r\left[\sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 4r\frac{\pi}{2} = 2\pi r$$

**Length of the arc of the parabola,** $y^2 = 4ax$

Here $$\frac{dy}{dx} = \frac{2a}{y}$$

and by formula (A)

$$s = \int\sqrt{\left(1 + \frac{4a^2}{y^2}\right)}dx = \int\sqrt{\left(1 + \frac{a}{x}\right)}dx \quad (a)$$

Let $$x = z^2 \quad \therefore dx = 2zdz$$

and $(a)$ becomes

$$s = 2\int\sqrt{(z^2 + a)}dz$$

$$= 2\left\{\frac{z\sqrt{(z^2 + a)}}{2} + \frac{a}{2}\log\left(z + \sqrt{z^2 + a}\right)\right\} + C$$

$$\therefore s = \sqrt{x}\sqrt{x + a} + a\log\left(\sqrt{x} + \sqrt{x + a}\right) + C$$

Suppose the length of the arc, from the vertex to the end of the latus-rectum, is required, we must integrate between the limits $x = 0$ and $x = a$

$$\therefore s = \sqrt{a}\sqrt{a + a} + a\log\left(\sqrt{a} + \sqrt{a + a}\right) - a\log\sqrt{a}$$

that is, $$s = a\sqrt{2} + a\log\left(1 + \sqrt{2}\right)$$

*Example.*—Find the length of the cables of a suspension

bridge 200 feet span 20 feet dip, assuming the cables to hang in parabolic arcs. Taking the origin at the lowest point and the axis of X vertical, the equation to the curve is

$$y^2 = 4ax\text{, but } x = 20 \text{ when } y = 100$$
$$\therefore 100^2 = 4a \times 20$$
$$\therefore a = 125$$

$$\therefore \text{Half the length} = \left[\sqrt{x(x+a)} + a\log\frac{\sqrt{x}+\sqrt{x+a}}{\sqrt{a}}\right]_0^{20}$$
$$= \sqrt{20 \times 145} + 125\log\frac{\sqrt{20}+\sqrt{145}}{\sqrt{125}}$$
$$= 103{\cdot}06 \text{ feet nearly}$$
$$\therefore \text{whole length} = 206{\cdot}12 \quad ,,$$

**Length of an arc of the catenary**

$$y = \frac{a}{2}\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}$$

Here
$$\frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}$$

therefore

$$s = \int\sqrt{\left[1 + \frac{1}{4}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}^2\right]}dx = \frac{1}{2}\int\left\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right\}dx$$
$$\therefore s = \frac{a}{2}\left\{e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right\}$$

The length of the arc from the vertex to the point $x = a$ is therefore

$$s = \frac{a}{2}(e - e^{-1})$$

**Length of the cycloid**

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta)$$

Here $$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

and $$dx = a(1-\cos\theta)d\theta$$

Substituting in (A) we have

$$s = a\int\sqrt{\left\{1+\frac{\sin^2\theta}{(1-\cos\theta)^2}\right\}}(1-\cos\theta)d\theta$$

that is,

$$s = a\sqrt{2}\int\sqrt{1-\cos\theta}\,d\theta = 2a\int\sin\tfrac{1}{2}\theta\,d\theta$$

$$\therefore s = 2a\Big[-2\cos\tfrac{1}{2}\theta\Big]_0^{2\pi}$$

for the whole length of the curve; therefore

$$s = 8a$$

That is, four times the diameter of the generating circle.

*Examples.*

1. Find the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = r^{\frac{2}{3}}$

$$Ans.\ s = \frac{3}{2}r^{\frac{1}{3}}x^{\frac{2}{3}} + c$$

2. Find the length of the curve whose equation is $ay^2 = x^3$

$$Ans.\ s = \frac{1}{\sqrt{a}}\left\{\frac{4a}{9}+x\right\}^{\frac{3}{2}} + c$$

3. Find the length of the arc of the logarithmic curve $y = e^x$

$$Ans.\ s = \log\frac{y}{1+\sqrt{1+y^2}} + \sqrt{1+y^2} + C$$

## CHAPTER XVIII

# EXPANSION OF PERIODIC FUNCTIONS

### Fourier's Theorem

A PERIODIC function is one which has the same value in every respect at equal intervals of time.

Assuming that the eccentric of a steam engine revolves with a uniform angular velocity, then the slide-valve is in the same position and is moving in the same direction with the same velocity at equal intervals of time. This interval of time, being the time taken for the eccentric to make one complete revolution, is called the *periodic time*. The algebraic expression for a periodic function is

$$f(t) = f(t + n\mathrm{T})$$

where T is the periodic time and $n$ is an integer.

Any periodic function of time may be expressed in the form—

$$\begin{aligned} f(t) = \; & \mathrm{A}_0 + \mathrm{A}_1 \sin pt + \mathrm{A}_2 \sin 2pt + \mathrm{A}_3 \sin 3pt \ldots \\ & + \mathrm{B}_1 \cos pt + \mathrm{B}_2 \cos 2pt + \mathrm{B}_3 \cos 3pt \;. \quad (1) \end{aligned}$$

where $\mathrm{A}_0$, $\mathrm{A}_1$, $\mathrm{A}_2$, . . .
$\mathrm{B}_1$, $\mathrm{B}_2$, $\mathrm{B}_3$ . . . etc. are constants which are to be found for any particular function of time, and $p$ is $\frac{2\pi}{\mathrm{T}}$ or $2\pi f$ where $f$ stands for the frequency.

It is important to remember that

$$\int_0^T \sin nptdt = 0$$

$$\int_0^T \cos nptdt = 0$$

$$\int_0^T \sin npt \sin mptdt = 0$$

$$\int_0^T \cos npt \cos mptdt = 0$$

$$\int_0^T \sin npt \cos mptdt = 0$$

$$\int_0^T \sin^2 nptdt = \frac{T}{2}$$

$$\int_0^T \cos^2 nptdt = \frac{T}{2}$$

where $n$ and $m$ are integers and T is the periodic time.

In order to obtain $A_0$ multiply every term of (1) by $dt$ and integrate between the limits 0 and T We have

$$\int_0^T f(t)dt = A_0T + A_1\int_0^T \sin ptdt + \text{etc.}$$
$$+ B_1\int_0^T \cos ptdt + \text{etc.}$$

all the terms on the right-hand except the first vanish

$$\therefore A_0 = \frac{1}{T}\int_0^T f(t)dt$$

That is $A_0$ is the average height of the original curve.

To get $A_n$ multiply every term of (1) by sin $nptdt$ and integrate between the limits 0 and T We have

$$\int_0^T f(t) \sin nptdt = A_n\int_0^T \sin^2 nptdt = A_n\frac{T}{2}$$

since all the terms on the right vanish except the term which contains $A_n$

$$\therefore A_n = \frac{2}{T}\int_0^T f(t)\sin nptdt \quad . \quad . \quad (2)$$

That is $A_n$ is twice the average height of the curve formed by multiplying together the corresponding ordinates of the original curve and $\sin npt$ for one complete period of the original curve.

Similarly $$B_n = \frac{2}{T}\int_0^T f(t)\cos nptdt \quad . \quad . \quad . \quad (3)$$

To get $A_1$, $A_2$, etc. substitute 1, 2, etc. respectively in (2) instead of $n$

To get $B_1$, $B_2$, $B_3$, etc. substitute 1, 2, 3, etc. in (3) instead of $n$

Suppose we require an E.M.F. to rise instantaneously from zero to a maximum $E_1$ and remain so for half a

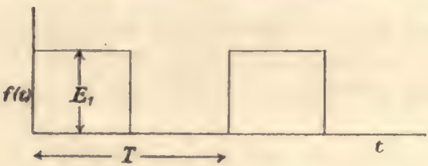


FIG. 57.

second, and then drop instantaneously to zero and remain so for the next half second, and let this operation be repeated every second.

Here E is a function of time, $f(t) = E_1$ from 0 to $\frac{T}{2}$ and $f(t) = 0$ from $\frac{T}{2}$ to T

$$A_0 = \frac{1}{T}\int_0^{\frac{T}{2}} E_1 dt = \frac{E_1}{2}$$

$$A_1 = \frac{2}{T}\int_0^{\frac{T}{2}} E_1 \sin ptdt = \frac{2E_1}{T}\left[-\frac{1}{p}\cos pt\right]_0^{\frac{T}{2}}$$

but $p = \dfrac{2\pi}{T}$

$$\therefore A_1 = \frac{2E_1}{T}\left[-\frac{T}{2\pi}\cos\frac{2\pi}{T}t\right]_0^{\frac{T}{2}} = \frac{2E_1}{\pi}$$

$$A_2 = \frac{2E_1}{T}\left[-\frac{T}{4\pi}\cos\frac{4\pi}{T}t\right]_0^{\frac{T}{2}} = 0$$

$$A_3 = \frac{2E_1}{T}\left[-\frac{T}{6\pi}\cos\frac{6\pi}{T}t\right]_0^{\frac{T}{2}} = \frac{2E_1}{3\pi}$$

Similarly $A_4 = 0$ and $A_5 = \dfrac{2E_1}{5\pi}$

$$B_1 = \frac{2E_1}{T}\int_0^{\frac{T}{2}}\cos ptdt = \frac{2E_1}{T}\left[\frac{T}{2\pi}\sin\frac{2\pi}{T}t\right]_9^{\frac{T}{2}} = 0$$

and $$B_2 = \frac{2E_1}{T}\left[\frac{T}{4\pi}\sin\frac{4\pi}{T}t\right]_0^{\frac{T}{2}} = 0$$

$$\therefore B_3 = 0 \text{ and } B_4 = 0 \text{ and so on}$$

$$\therefore f(t) = \frac{E_1}{2} + \frac{2E_1}{\pi}\left\{\sin pt + \frac{1}{3}\sin 3pt + \frac{1}{5}\sin 5pt + \text{etc.}\right\}$$

That is, the required E.M.F. may be obtained by a constant E.M.F. of $\dfrac{E_1}{2}$ and an alternating E.M.F. of maximum value $\dfrac{2E_1}{\pi}$ of the same frequency as the required E.M.F. and an E.M.F. of maximum value $\dfrac{2E_1}{3\pi}$ and frequency three times that of the required E.M.F. and so on to infinity. If, however, only a few of the first terms be taken, we get approximately the required E.M.F.

*Example.*—Suppose we require an E.M.F. to rise gradually from zero to $E_1$ during the period and then drop instantaneously to zero.

Here $A_0 = \frac{1}{T}\int_0^T f(t)dt$

Now $f(t) = kt$ $k$ being a constant $= \frac{E_1}{T}$

$$\therefore f(t) = \frac{E_1 t}{T}$$

Therefore $A_0 = \frac{E_1}{T^2}\int_0^T tdt = \frac{E_1}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{E_1}{2}$

$$A_n = \frac{2E_1}{T^2}\int_0^T t \sin nptdt$$

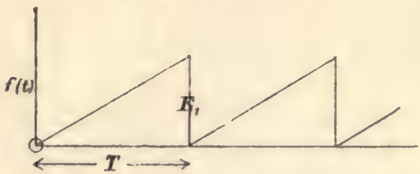


FIG. 58.

On integrating by parts, we have

$$A_n = \frac{2E_1}{T^2}\left[-\frac{t}{np}\cos npt + \frac{1}{(np)^2}\sin npt\right]_0^T$$

Let $n$ be replaced by 1, 2, 3, etc. and we obtai $A_1$ $A_2$ $A_3$ etc. respectively.

$$\therefore A_1 = \frac{2E_1}{T^2}\left[-\frac{T^2}{2\pi}\right] = -\frac{E_1}{\pi}$$

and $$A_2 = \frac{2E_1}{T^2}\left[-\frac{T^2}{4\pi}\right] = -\frac{E_1}{2\pi}$$

Similarly $A_3 = -\frac{E_1}{3\pi}$ and $A_4 = -\frac{E_1}{4\pi}$ etc.

Again $B_n = \frac{2E_1}{T^2}\int_0^T t \cos nptdt$

and on integrating by parts, we have

$$B_n = \frac{2E_1}{T^2}\left[\frac{Tt}{2\pi}\sin npt + \frac{1}{(np)^2}\cos npt\right]_0^T$$

Let $n$ be replaced by 1, 2, 3, etc., and we get $B_1$ $B_2$ $B_3$ etc. respectively.

$$\therefore B_1 = 0 \text{ and } B_2 = 0 \text{ etc.}$$

$$\therefore f(t) = \frac{E_1}{2} - \frac{E_1}{\pi}\left\{\sin pt + \frac{1}{2}\sin 2pt + \frac{1}{3}\sin 3pt \text{ etc.}\right\}$$

Develop the periodic function, Fig. 59.

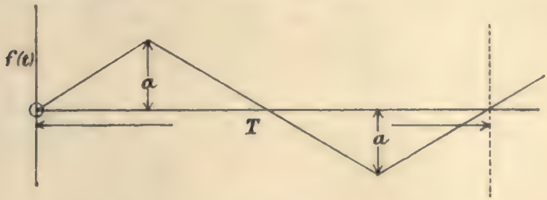


FIG. 59.

Here $$A_0 = \frac{1}{T}\int_0^T f(t)dt$$

= average height of $f(t)$ which is zero;

$$A_0 = 0$$

Again, $$A_1 = \frac{2}{T}\int f(t)\sin\frac{2\pi t}{T}dt$$

Now $f(t) = Kt$ between the limits $-\frac{T}{4}$ and $+\frac{T}{4}$ and

$f(t) = a$ when $t = \frac{T}{4}$

$$\therefore a = K\frac{T}{4}$$

$$\therefore K = \frac{4a}{T}$$

Again, we have $f(t) = 2a\left(1 - \frac{2t}{T}\right)$ between the limits $= \frac{T}{4}$ and $t = \frac{3T}{4}$ therefore

$$A_1 = \frac{2}{T}\int_{-\frac{T}{4}}^{+\frac{T}{4}} \frac{4at}{T} \sin \frac{2\pi t}{T} dt + \frac{2}{T}\int_{\frac{T}{4}}^{\frac{3T}{4}} 2a\left(1 - \frac{2t}{T}\right) \sin \frac{2\pi t}{T} dt$$

$$= \frac{8a}{T^2}\int_{-\frac{T}{4}}^{+\frac{T}{4}} t \sin \frac{2\pi t}{T} dt + \frac{4a}{T}\int_{\frac{T}{4}}^{+\frac{3T}{4}} \sin \frac{2\pi t}{T} dt$$

$$- \frac{8a}{T^2}\int_{\frac{T}{4}}^{+\frac{3T}{4}} t \sin \frac{2\pi t}{T} dt$$

$$\therefore A_1 = \frac{8a}{T^2}\left[-\frac{Tt}{2\pi} \cos \frac{2\pi t}{T} + \left(\frac{T}{2\pi}\right)^2 \sin \frac{2\pi t}{T}\right]_{-\frac{T}{4}}^{\frac{T}{4}}$$

$$+ \frac{4a}{T}\left[-\frac{T}{2\pi} \cos \frac{2\pi t}{T}\right]_{\frac{T}{4}}^{\frac{3T}{4}}$$

$$- \frac{8a}{T^2}\left[-\frac{Tt}{2\pi} \cos \frac{2\pi t}{T} + \left(\frac{T}{2\pi}\right)^2 \sin \frac{2\pi t}{T}\right]_{\frac{T}{4}}^{\frac{3T}{4}}$$

$$\therefore A_1 = \frac{8a}{\pi^2}$$

Similarly, we find that

$$B_1 = 0, \quad B_2 = 0, \quad B_3 = 0, \text{ etc.,}$$
$$A_2 = 0, \quad A_4 = 0, \quad A_6 = 0, \text{ etc.,}$$

and

$$A_3 = -\frac{8a}{9\pi^2}, \quad A_5 = \frac{8a}{25\pi^2}, \quad A_7 = -\frac{8a}{49\pi^2}, \text{ etc.}$$

$$\therefore f(t) = \frac{8a}{\pi^2}\left\{\sin \frac{2\pi t}{T} - \frac{1}{9} \sin \frac{6\pi t}{T} + \frac{1}{25} \sin \frac{10\pi t}{T}\right.$$

$$\left. - \frac{1}{49} \sin \frac{14\pi t}{T} + \text{etc.}\right\}$$

*Examples.*

1. Develop

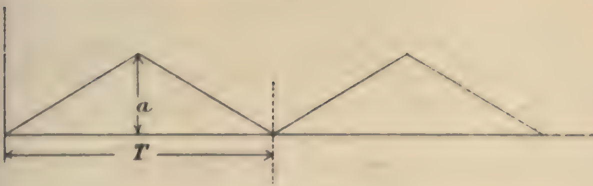


FIG. 60.

2. Develop


FIG. 61.

3. Develop

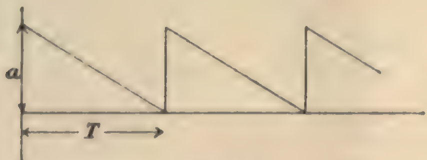


FIG. 62.

Development of

$$1.\ f(t) = \frac{a}{2} - \frac{4a}{\pi^2}\left\{\cos\frac{2\pi t}{T} + \frac{1}{9}\cos\frac{6\pi t}{T} + \frac{1}{25}\cos\frac{10\pi t}{T} + \frac{1}{49}\cos\frac{14\pi t}{T} + \text{etc.}\right\}$$

$$2.\ f(t) = \frac{4a}{\pi}\left\{\sin\frac{2\pi t}{T} + \frac{1}{3}\sin\frac{6\pi t}{T} + \frac{1}{5}\sin\frac{10\pi t}{T} + \text{etc.}\right.$$

$$3.\ f(t) = \frac{a}{2} + \frac{a}{\pi}\left\{\sin\frac{2\pi t}{T} + \frac{1}{2}\sin\frac{4\pi t}{T} + \frac{1}{3}\sin\frac{6\pi t}{T} + \text{etc.}\right\}$$

It is possible to tell by inspection when a periodic function is built up of either sine functions or cosine functions or both sines and cosines.

If the time axis be taken at the mean height of the curve that is where $A_0$ would be zero, and if ordinates to the curve be taken equally distant from the ends of a complete period then if the ordinates are equal and opposite in sign, the function contains sine terms only.

If the ordinates are equal and of the same sign the function contains cosine terms only.


FIG. 63.

If the ordinates are not equal the function consists of both sines and cosines.

If a periodic function contains the odd harmonics only the first half period is exactly similar to the second half period. (See Fig. 63.)

There are several other methods of developing a periodic function, but the only one which space permits is that which has been simplified by Dr. S. P. Thompson, F.R.S., and which is fully explained in his book on "Dynamo Machinery," vol. ii. chap. ii.

Assuming that there are no even harmonics, the following is his method of developing a periodic function up to the 11th harmonic,

If the even harmonics be not present, then the two half periods will be similar, and if the time axis be taken midway between the highest and lowest points, there will be no constant term.

Divide the half period into 12 equal parts and erect ordinates at each division. Let the ordinates be denoted by $y_0\ y_1\ y_2\ y_3\ \cdots\ y_{10}\ y_{11}\ y_{12}$ Then $y_0$ and $y_{12}$ will be zero.

Arrange the ordinates as follows

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |
| | $y_{12}$ | $y_{11}$ | $y_{10}$ | $y_9$ | $y_8$ | $y_7$ | |
| adding | | $s_1$ | $s_2$ | $s_3$ | $s_4$ | $s_5$ | $s_6$ |
| subtracting | | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |

$s_1 = y_1 + y_{11}$ and so on, and $d_1 = y_1 - y_{11}$ and so on. Special attention must be given to the signs in all cases.

On grouping the numbers in order to obtain values to be used with the 3rd and 9th harmonics we have

$$s_1 + s_3 - s_5 = r_1 \text{ say}$$
$$s_2 - s_6 = r_2$$
$$d_1 - d_3 - d_5 = e_1$$

Now take the numerical values of $s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ r_1\ d_1$ etc. and substitute in the table below, after multiplying each by the sine of the angle placed opposite in the left-hand column.

| Angle. | Sine Terms. | | | Cosine Terms. | | |
|---|---|---|---|---|---|---|
| | 1st and 11th Harmonics. | 3rd and 9th Harmonics. | 5th and 7th Harmonics. | 1st and 11th Harmonics. | 3rd and 9th Harmonics. | 5th and 7th Harmonics. |
| Sin 15° = ·259 | $s_1$ | | $s_5$ | $d_5$ | | $d$ |
| Sin 30° = ·500 | $s_2$ | | $s_2$ | $d_4$ | | $d_4$ |
| Sin 45° = ·707 | $s_3$ | $r_1$ | $-s_3$ | $d_3$ | $e_1$ | $-d$ |
| Sin 60° = ·866 | $s_4$ | | $-s_4$ | $d_2$ | | $-d_2$ |
| Sin 75° = ·966 | $s_5$ | | $s_1$ | $d_1$ | | $d$ |
| Sin 90° = 1·00 | $s_6$ | $r_2$ | $s_6$ | — | $-d_4$ | |
| Total 1st col. | | | | | | |
| ,, 2nd ,, | | | | | | |
| Sum . . . . | $6A_1$ | $6A_3$ | $6A_5$ | $6B_1$ | $6B_3$ | $6B_5$ |
| Difference . . | $6A_{11}$ | $6A_9$ | $6A_7$ | $6B_{11}$ | $6B_9$ | $6B_7$ |

Then

$$\begin{aligned} y = \ & A_1 \sin pt + A_3 \sin 3pt + A_5 \sin 5pt + A_7 \sin 7pt \\ & + A_9 \sin 9pt + A_{11} \sin 11pt + \ldots \\ & + B_1 \cos pt + B_3 \cos 3pt + B_5 \cos 5pt + B_7 \cos 7pt \\ & + B_9 \cos 9pt + B_{11} \cos 11pt + \ldots \end{aligned}$$

*Example.*—Given that the 12 ordinates are, 0 , 20 , 30 , 35 , 40 , 38 , 32 , 26 , 22 , 18 , 16 , and 12

Arranging them thus

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0 | 20 | 30 | 35 | 40 | 38 | 32 |
| | 0 | 12 | 16 | 18 | 22 | 26 | |
| adding | | 32 | 46 | 53 | 62 | 64 | 32 |
| subtracting | | 8 | 14 | 17 | 18 | 12 | 32 |

Grouping we have

$$\begin{aligned} 32 + 53 \quad - 64 &= 21 \\ 46 - 32 \quad &= 14 \\ 8 - 17 \quad - 12 &= -21 \end{aligned}$$

Entering in the table after multiplying by the sines opposite thus

| Angle | Sine Terms | | | | | | Cosine Terms. | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1st and 11th Harmonics. | | 3rd and 9th Harmonics. | | 5th and 7th Harmonics. | | 1st and 11th Harmonics. | | 3rd and 9th Harmonics. | | 5th and 7th Harmonics. | |
| Sin 15° = ·259 | 8·29 | | | | 16·5 | | | 3·11 | | | | 2·07 |
| Sin 30° = ·500 | | 23 | | | | 23 | 9 | | | | 9 | |
| Sin 45° = ·707 | 37·5 | | 14·82 | | −37·4 | | | 12 | | −14·8 | | −12 |
| Sin 60° = ·866 | | 53·6 | | | | −53·6 | 12·1 | | | | −12·1 | |
| Sin 75° = ·966 | 61·8 | | | | 30·9 | | | 7·73 | | | | 11·6 |
| Sin 90° = 1·000 | | 32 | | 14 | | 32 | | | −18 | | | |
| Total 1st col. | 107·59 | | 14·82 | | 10· | | 21·1 | | −18 | | −3·1 | |
| " 2nd " | | 108·6 | | 14 | | 1·4 | | 22·84 | | −14·8 | | 1·67 |
| Sum . . . | $216{\cdot}19 = 6A_1$ | | $28{\cdot}82 = 6A_3$ | | $11{\cdot}4 = 6A_5$ | | $43{\cdot}94 = 6B_1$ | | $-32{\cdot}8 = 6B_3$ | | $-1{\cdot}44 = 6B_5$ | |
| Difference . | $-1{\cdot}01 = 6A_{11}$ | | $\cdot 82 = 6A_9$ | | $8{\cdot}6 = 6A_7$ | | $-1{\cdot}74 = 6B_{11}$ | | $-3{\cdot}2 = 6B_9$ | | $-4{\cdot}77 = 6B_7$ | |

$$\therefore A_1 = 36{\cdot}03, \quad A_3 = 4{\cdot}8, \quad A_5 = 1{\cdot}9, \quad A_7 = 1{\cdot}43$$

$$A_9 = \cdot 136, \quad A_{11} = -\cdot 17.$$

$$B_1 = 7{\cdot}32, \quad B_3 = -5{\cdot}46, \quad B_5 = -\cdot 24,$$

$$B_7 = -\cdot 8, \quad B_9 = -\cdot 53, \quad B_{11} = -\cdot 29.$$

$$\therefore y = 36{\cdot}03 \sin x + 4{\cdot}8 \sin 3x + 1{\cdot}9 \sin 5x + 1{\cdot}43 \sin 7x + \cdot 136 \sin 9x - \cdot 17 \sin 11x + 7{\cdot}32 \cos x - 5{\cdot}46 \cos 3x - \cdot 24 \cos 5x - \cdot 8 \cos 7x - \cdot 53 \cos 9x - \cdot 29 \cos 11x$$

Reasons for the method of grouping

$$A_n = \frac{2}{T}\int_0^T f(t) \sin \frac{2\pi}{T} nt\,dt$$

Taking 12 ordinates the last one being zero we have

$$A_1 = \tfrac{2}{12}\{y_1 \sin 15° + y_2 \sin 30° + y_3 \sin 45° + y_4 \sin 60° + y_5 \sin 75° + y_6 \sin 90° + y_7 \sin 75° + y_8 \sin 60° + y_9 \sin 45° + y_{10} \sin 30° + y_{11} \sin 15°\}$$

$$\therefore A_1 = \tfrac{1}{6}\{(y_1 + y_{11}) \sin 15° + (y_2 + y_{10}) \sin 30° + (y_3 + y_9) \sin 45° + (y_4 + y_8) \sin 60° + (y_5 + y_7) \sin 75° + y_6 \sin 90°\}$$

$$= \tfrac{1}{6}\{s_1 \sin 15° + s_2 \sin 30° + s_3 \sin 45° + s_4 \sin 60° + s_5 \sin 75° + s_6 \sin 90°\}$$

Similarly

$$A_{11} = \tfrac{1}{6}\{s_1 \sin 15^\circ - s_2 \sin 30^\circ + s_3 \sin 45^\circ - s_4 \sin 60^\circ + s_5 \sin 75^\circ - s_6 \sin 90^\circ\}$$

$$A_3 = \tfrac{1}{6}\{(s_1 + s_3 - s_5) \sin 45^\circ + (s_2 - s_6) \sin 90^\circ\} = \tfrac{1}{6}\{r_1 \sin 45^\circ + r_2 \sin 90^\circ\}$$

$$A_9 = \tfrac{1}{6}\{(s_1 + s_3 - s_5) \sin 45^\circ - (s_2 - s_6) \sin 90^\circ\} = \tfrac{1}{6}\{r_1 \sin 45^\circ - r_2 \sin 90^\circ\}$$

$$A_5 = \tfrac{1}{6}\{s_1 \sin 75^\circ + s_2 \sin 30^\circ - s_3 \sin 45^\circ - s_4 \sin 60^\circ + s_5 \sin 75^\circ + s_6 \sin 90^\circ\}$$

$$A_7 = \tfrac{1}{6}\{s_1 \sin 75^\circ - s_2 \sin 30^\circ - s_3 \sin 45^\circ + s_4 \sin 60^\circ + s_5 \sin 75^\circ - s_6 \sin 90^\circ\}$$

Similarly

$$B_1 = \tfrac{1}{6}\{d_1 \sin 75^\circ + d_2 \sin 60^\circ + d_3 \sin 45^\circ + d_4 \sin 30^\circ + d_5 \sin 15^\circ\}$$

$$B_{11} = \tfrac{1}{6}\{- d_1 \sin 75^\circ + d_2 \sin 60^\circ - d_3 \sin 45^\circ + d_4 \sin 30^\circ - d_5 \sin 15^\circ\}$$

$$B_3 = \tfrac{1}{6}\{(d_1 - d_3 - d_5) \sin 45^\circ - d_4 \sin 90^\circ\}$$

$$B_9 = \tfrac{1}{6}\{-(d_1 - d_3 - d_5) \sin 45^\circ - d_4 \sin 90^\circ\}$$

$$B_5 = \tfrac{1}{6}\{d_1 \sin 15^\circ + d_4 \sin 30^\circ - d_3 \sin 45^\circ - d_2 \sin 60^\circ + d_5 \sin 75^\circ\}$$

$$B_7 = \tfrac{1}{6}\{-d_1 \sin 15^\circ + d_4 \sin 30^\circ + d_3 \sin 45^\circ - d_2 \sin 60^\circ - d_5 \sin 75^\circ\}$$

CHAPTER XIX

## CENTRE OF A PLANE AREA AND CENTRE OF MASS

THE sum of the moments of all the elements of any plane area about a line in its plane is equal to the moment of the whole area about the same line.

In the accompanying figure it is required to find the

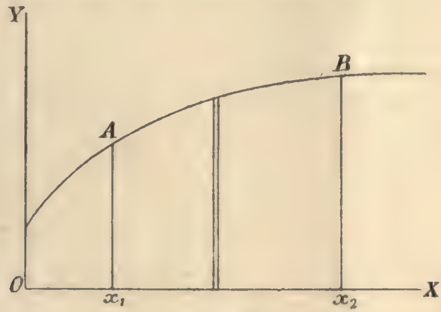


FIG. 64.

centre of the area $Ax_1x_2B$ Let the co-ordinates of the centre of area be $\bar{x}$ and $\bar{y}$

Suppose the area to be split up into an indefinite number of strips parallel to the axis of Y the breadth of each being $dx$ Let the height of any one of these strips, at a distance $x$ from the axis of Y be $y$ then the area of this strip will be $ydx$ The whole area

$$Ax_1x_2B = \int_{x_1}^{x_2} ydx$$

The moment of an element of area about the axis of Y is

$$xydx$$

Let $\bar{x}$ and $\bar{y}$ be the co-ordinates of the centre of area, then the moment of the whole area about the axis of Y is

$$\bar{x}\int_{x_1}^{x_2} ydx\,,$$

and this must be equal to

$$\int_{x_1}^{x_2} xydx$$

$$\therefore\ \bar{x} = \frac{\int_{x_1}^{x_2} xydx}{\int_{x_1}^{x_2} ydx} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the general formula for finding the $\bar{x}$ of the centre of an area enclosed by the axis of X the curve $y = f(x)$ and the two ordinates at the points $x_1$ and $x_2$ when rectangular co-ordinates are used.

To get the $\bar{y}$ of the centre of area, take moments about the axis of X The distance of the centre of a strip from the axis of X is $\frac{y}{2}$ therefore the moment of the strip $ydx$ about the axis of X is $\frac{1}{2}y^2dx$

$$\therefore\ \bar{y} = \frac{\frac{1}{2}\int_{x_1}^{x_2} y^2dx}{\int_{x_1}^{x_2} ydx} \quad . \quad . \quad . \quad . \quad (2)$$

This is the general formula for getting the $\bar{y}$ of the centre of any plane area contained by $y = f(x)$ the axis of X and the ordinates at $x_1$ and $x_2$

If we require the centre of area bounded by the curve $x = f(y)$ the axis of Y and two horizontal lines parallel to the axis of X at the points $y_1$ and $y_2$

then $$\bar{x} = \frac{\frac{1}{2}\int_{y_1}^{y_2} x^2dy}{\int_{y_1}^{y_2} xdy}$$

and
$$\bar{y} = \frac{\int_{y_1}^{y_2} xy\,dy}{\int_{y_1}^{y_2} x\,dy}$$

*Example.*—Centre of area of the parabola

$$y^2 = 4ax$$

between the limits $x = 0$ and $x = x_1$

Here $\bar{x} = \dfrac{\int_0^{x_1} xy\,dx}{\int_0^{x_1} y\,dx}$ $\qquad y = 2\sqrt{a}x^{\frac{1}{2}}$

$$\therefore \bar{x} = \frac{2\sqrt{a}\int_0^{x_1} x^{\frac{3}{2}}dx}{2\sqrt{a}\int x^{\frac{1}{2}}dx} = \frac{\left[\frac{2}{5}x^{\frac{5}{2}}\right]_0^{x_1}}{\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^{x_1}} = \frac{3}{5}x_1$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^{x_1} y^2dx}{\int_0^{x_1} y\,dx} = \frac{2a\int_0^{x_1} x\,dx}{2\sqrt{a}\int_0^{x_1} x^{\frac{1}{2}}dx} = \sqrt{a}\frac{\left[\frac{x^2}{2}\right]_0^{x_1}}{\left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^{x_1}}$$

$$= \frac{3}{4}\sqrt{a}x_1^{\frac{1}{2}} = \frac{3}{8}\times 2\sqrt{a}x_1^{\frac{1}{2}} = \frac{3}{8}y_1$$

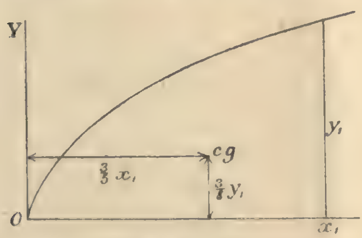


FIG. 65.

The centre of area of a segment of a parabola is at the point $\frac{3}{5}x_1 \quad \frac{3}{8}y_1$

**Polar co-ordinates.**

If the equation of the curve be given in polar co-ordinates where the origin is the pole and the axis of X is the initial line, and we require the $\bar{x}$ and $\bar{y}$ of the area OAB

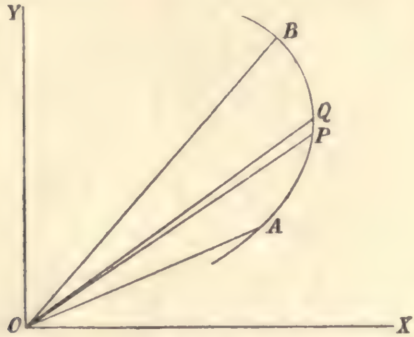


FIG. 66.

Imagine the area OAB divided up into an indefinite number of triangles whose common vertex is O one of them being OPQ The area of OPQ is $\frac{1}{2}r^2d\theta$ where $r = $ OP and $\theta$ is the vectorial angle of P That is the angle XOP

The $\bar{x}$ of the centre of area OPQ is $\frac{2}{3}r\cos\theta$ and the $\bar{y}$ of it is $\frac{2}{3}r\sin\theta$

$$\text{Hence} \qquad \bar{x} = \frac{\int\frac{2}{3}r\cos\theta\frac{1}{2}r^2d\theta}{\int\frac{1}{2}r^2d\theta}$$

$$\text{that is,} \qquad \bar{x} = \frac{\frac{2}{3}\int r^3\cos\theta d\theta}{\int r^2d\theta} \quad . \quad . \quad . \quad . \quad (3)$$

between the proper limits.

$$\text{Similarly} \qquad \bar{y} = \frac{\frac{2}{3}\int r^3\sin\theta d\theta}{\int r^2d\theta} \quad . \quad . \quad . \quad . \quad (4)$$

The centre of any plane curve whose polar equation is given may be obtained by (3) and (4).

*Example.*—To find the centre of area of a quadrant of a

circle. Taking the pole at the centre of the circle the radius vector is constant and equal to the radius of the circle

$$\therefore \bar{x} = \frac{\frac{2}{3}r^3\int_0^{\frac{\pi}{2}} \cos\theta d\theta}{r^2\int_0^{\frac{\pi}{2}} d\theta} = \frac{2}{3}r\frac{\left[\sin\theta\right]_0^{\frac{\pi}{2}}}{\left[\theta\right]_0^{\frac{\pi}{2}}}$$

$$= \frac{2}{3}r\frac{1}{\frac{\pi}{2}} = \frac{4r}{3\pi}$$

Similarly $\bar{y} = \dfrac{4r}{3\pi}$

The centre of area of a semicircle lies at $\dfrac{4r}{3\pi}$ from the centre on the axis of symmetry.

**Centre of gravity of an arc of a plane curve.**

Let AB be an arc of any plane curve, and let $ds$ be the length of an element of the arc.

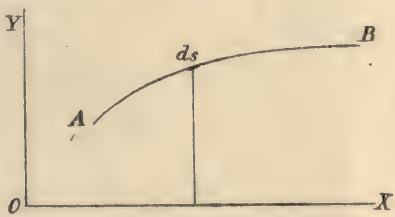


FIG. 67.

Then $yds$ is the moment of the element of arc about the axis of X

Let $\bar{y}$ be the centre of gravity of the arc AB

then $$\bar{y}\int ds = \int yds$$

between the proper limits.

Now $$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

$$\therefore \bar{y} = \frac{\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx}{\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx}$$

Similarly $$\bar{x} = \frac{\int x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx}{\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx}$$

*Example.*—Centre of gravity of an arc of a circle subtending an angle $2\alpha$ at the centre.

Taking the centre as origin, we have $ds = rd\theta$

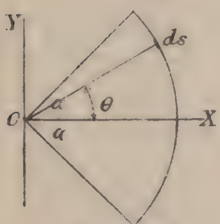


FIG. 68.

$$\bar{x}\int ds = r\int_{-\alpha}^{+\alpha}\cos\theta ds$$

$$\therefore \bar{x} = \frac{r^2\int_{-\alpha}^{+\alpha}\cos\theta d\theta}{r\int_{-\alpha}^{+\alpha}d\theta}$$

$$= r\frac{\Big[\sin\theta\Big]_{-\alpha}^{+\alpha}}{\Big[\theta\Big]_{-\alpha}^{+\alpha}} = \frac{r\sin\alpha}{\alpha}$$

$$\bar{y} = 0$$

If $2\alpha = \pi$ then $\bar{x} = \dfrac{2r}{\pi}$

**Centre of mass of solid of revolution.**

Let the solid be generated by the revolution of a curve or straight line about the axis of X Any section made by a plane perpendicular to the axis of X will be circular. Let $y$ be the radius of any section at a distance $x$ from the origin.

An element of volume will be $\pi y^2 dx$ where $dx$ is the thickness of the disc.

The moment of this element about the origin will be

$$\pi x y^2 dx$$

Hence $$\bar{x} = \frac{\int \pi x y^2 dx}{\int \pi y^2 dx}$$

that is, $$\bar{x} = \frac{\int x y^2 dx}{\int y^2 dx} \quad \ldots \ldots \quad (1)$$

*Example.*—Find the centre of mass of a cone whose height is $h$ and radius of its base is $r$

Taking the vertex as origin and axis as the axis of X the radius of a right section at a distance $x$ from the origin is

$$\frac{rx}{h} = y$$

therefore $$\bar{x} = \frac{\int_0^h \frac{xr^2x^2}{h^2}dx}{\int_0^h \frac{r^2x^2}{h^2}dx} = \frac{\int_0^h x^3 dx}{\int_0^h x^2 dx}$$

Hence $$\bar{x} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h} = \frac{3}{4}h$$

**Mass centre of a hemisphere.**

Take the centre of the sphere as origin then

$$x^2 + y^2 = R^2$$

$$\bar{x} = \frac{\int_0^R x(R^2 - x^2)dx}{\int_0^R (R^2 - x^2)dx} = \frac{\left[\frac{R^2x^2}{2} - \frac{x^4}{4}\right]_0^R}{\left[R^2x - \frac{x^3}{3}\right]_0^R}$$

$$= \tfrac{3}{8}R \cdot$$

**Mass centre of a paraboloid of revolution.**

Here $y^2 = 4ax$

$$\bar{x} = \frac{4a\int_0^{x_1} x^2dx}{4a\int_0^{x_1} xdx} = \frac{\left[\frac{x^3}{3}\right]_0^{x_1}}{\left[\frac{x^2}{2}\right]_0^{x_1}} = \frac{2}{3}x_1$$

**Mass centre of a surface of revolution.**

Let AB an arc of a plane curve revolve about the axis of X An element of surface is $2\pi yds$

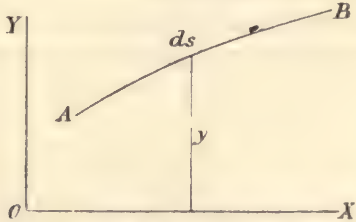


FIG. 69.

Let $\bar{x}$ be the distance of the centre of gravity of the surface from the origin, then

$$\bar{x}\int 2\pi yds = \int 2\pi xyds$$

$$\therefore \bar{x} = \frac{\int xyds}{\int yds} = \frac{\int xy\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx}{\int y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx}$$

*Example.*—Centre of area of a hemispherical shell (Fig. 70).

$$x^2 + y^2 = R^2$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}.$$

$$\bar{x} = \frac{\int xy\sqrt{\left(1+\frac{x^2}{y^2}\right)}dx}{\int y\sqrt{\left(1+\frac{x^2}{y^2}\right)}dx}$$

$$\therefore x = \frac{\int_0^R x\mathrm{R}dx}{\int_0^R \mathrm{R}dx} = \frac{\left[\frac{x^2}{2}\right]_0^R}{\left[x\right]_0^R} = \frac{\mathrm{R}}{2}$$

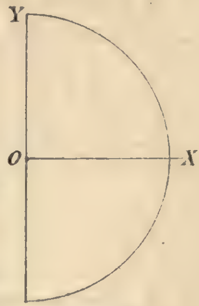

FIG. 70.

Centre of mass of a thin rod, the density being proportional to the cube of the distance from one end.

Let the density of the rod at unit distance from one end be $m$, therefore the density at $x$ from the end is $mx^3$

Let the centre of mass be $\bar{x}$ from the end

$$\text{then} \quad \bar{x}\int_0^l mx^3dx = \int_0^l mx^4dx$$

$$\therefore \bar{x} = \frac{\int_0^l x^4dx}{\int_0^l x^3dx} = \frac{\left[\frac{x^5}{5}\right]_0^l}{\left[\frac{x^4}{4}\right]_0^l} = \frac{4}{5}l$$

If the density varied as the $n$th power of the distance from one end, then the centre of mass would be at a distance of

$$\frac{(n+1)}{n+2}l$$

from the end.

*Examples.*

1. Find the centre of area enclosed by the curve the equation of which is

$$r = a(1+\cos\theta) \qquad \textit{Ans.} \quad \bar{x} = \tfrac{5}{6}a$$

2. Find the centre of area of a quadrant of an ellipse the equation of which is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \qquad Ans. \ \bar{x} = \frac{4a}{3\pi}.\bar{y} = \frac{4b}{3\pi}$$

3. Find the centre of area of the cycloid

$$x = R(\theta - \sin\theta)$$
$$y = R(1 - \cos\theta)$$

$$Ans. \ \bar{x} = \pi R.\bar{y} = \frac{5R}{6}$$

4. Find the volume generated by the revolution of the cycloid, mentioned in the previous example, about the axis of X *Ans.* $5\pi^2 R^3$

5. Find the centre of gravity of a zone of a sphere of height $h$ *Ans.* $\frac{h}{2}$ from base.

6. Find the surface formed by the revolution of a cycloid about its base. *Ans.* $\frac{64\pi a^2}{3}$

7. Find the volume generated by the revolution of a semi-parabola about the latus rectum

*Ans.* $\frac{16a^3}{15}$

## CHAPTER XX

# HYPERBOLIC FUNCTIONS

LET $r$ denote the radius, $\theta$ the angle at the centre and A the area of the sector XOP

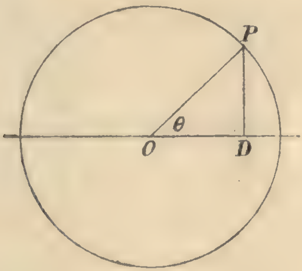


FIG. 71.

$$\text{Then} \qquad A = \tfrac{1}{2}r^2\theta$$

$$\therefore \theta = \frac{2A}{r^2}$$

$$\text{Hence} \qquad \cos\theta = \cos\frac{2A}{r^2}$$

$$\text{and} \qquad \sin\theta = \sin\frac{2A}{r^2}$$

and so for the other circular functions. The sense of the sector XOP is the same as that of the angle XOP

## Hyperbolic Functions

Let a point move along the curve from the vertex A of one branch of a *rectangular hyperbola* whose centre is O and semi-axis is $r$ to a position P Let A denote the area of the sector AOP and let $v = \frac{2A}{r^2}$ Where $v$ is the measure of the sector AOP the unit of area being the square, the diagonal of which is the semi-axis $r$

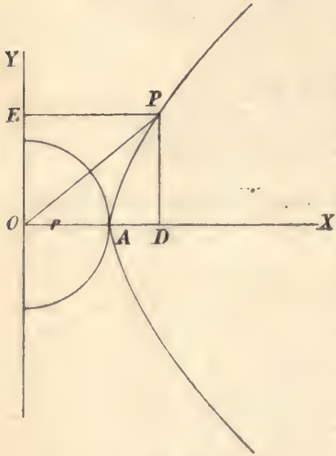


FIG. 72.

Let OE and OD be the vertical and horizontal projections of OP respectively

Then

$$\frac{OD}{OA} = \cosh v \text{ the hyperbolic cosine of } v$$

$$\therefore \frac{OD}{r} = \frac{x}{r} = \cosh v$$

$$\frac{OE}{r} = \frac{y}{r} = \sinh v$$

$$\frac{OE}{OD} = \frac{y}{x} = \tanh v$$

$$\frac{r}{OD} = \frac{r}{x} = \operatorname{sech} v$$

$$\frac{r}{OE} = \frac{r}{y} = \operatorname{cosech} v$$

$$\frac{OD}{OE} = \frac{x}{y} = \coth v$$

$$\text{The area of OPA} = \text{OPD} - \text{APD}$$
$$= \frac{1}{2}xy - \int_r^x y\,dx$$

but $y = \sqrt{x^2 - r^2}$ since the hyperbola is rectangular

$$\therefore \text{OPA} = \frac{1}{2}xy - \int_r^x \sqrt{(x^2 - r^2)}\,dx$$
$$= \frac{1}{2}xy - \left[\frac{x\sqrt{x^2-r^2}}{2} - \frac{r^2}{2}\log\left(x + \sqrt{x^2 - r^2}\right)\right]_r^x$$
$$= \frac{1}{2}xy - \left[\frac{xy}{2} - \frac{r^2}{2}\log\left(\frac{x+y}{r}\right)\right]$$
$$= \frac{r^2}{2}\log\left(\frac{x}{r} + \frac{y}{r}\right) = \frac{r^2}{2}v$$

$$\therefore \text{Area OPA} = \frac{r^2}{2}\log\left(\frac{x}{r} + \frac{y}{r}\right) = \frac{r^2}{2}v$$

$$\therefore \log\left(\frac{x}{r} + \frac{y}{r}\right) = v$$

$$\therefore \frac{x}{r} + \frac{y}{r} = e^v \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But $$\frac{x^2}{r^2} - \frac{y^2}{r^2} = \left(\frac{x}{r} - \frac{y}{r}\right)\left(\frac{x}{r} + \frac{y}{r}\right) = 1$$

$$\therefore \frac{x}{r} - \frac{y}{r} = e^{-v} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore \frac{x}{r} = \frac{e^v + e^{-v}}{2} = \cosh v$$

and $$\frac{y}{r} = \frac{e^v - e^{-v}}{2} = \sinh v$$

$$\text{Sech } v = \frac{1}{\cosh v}, \quad \tanh v = \frac{\sinh v}{\cosh v} = \frac{e^v - e^{-v}}{e^v + e^{-v}}$$

$$\text{cosech } v = \frac{1}{\sinh v}$$

$$\cosh^2 v - \sinh^2 v = \frac{x^2}{r^2} - \frac{y^2}{r^2} = 1$$

$$\text{sech}^2 v = \frac{4}{(e^v + e^{-v})^2} = 1 - \left(\frac{e^v - e^{-v}}{e^v + e^{-v}}\right)^2 = 1 - \tanh^2 v$$

$$\text{and} \quad \operatorname{cosech}^2 v = \frac{4}{(e^v - e^{-v})^2} = \left(\frac{e^v + e^{-v}}{e^v - e^{-v}}\right)^2 - 1$$
$$= \coth^2 v - 1$$

$$\therefore \cosh^2 v - \sinh^2 v = 1 \quad . \quad . \quad . \quad (3)$$
$$\operatorname{sech}^2 v = 1 - \tanh^2 v \quad . \quad . \quad (4)$$
$$\operatorname{cosech}^2 v = \coth^2 v - 1 \quad . \quad . \quad (5)$$

and
$$\cosh(-v) = \cosh v$$
$$\sinh(-v) = -\sinh v$$
$$\operatorname{sech}(-v) = \operatorname{sech} v$$
$$\tanh(-v) = -\tanh v$$
$$\operatorname{cosech}(-v) = -\operatorname{cosech} v$$
$$\coth(-v) = -\coth v$$

$$\operatorname{Sinh}(u+v) = \sinh u \cosh v + \cosh u \sinh v$$
$$\cosh(u+v) = \cosh u \cosh v + \sinh u \sinh v$$
$$\tanh(u+v) = \frac{\tanh u + \tanh v}{1 + \tanh u \tanh v}$$

$$\cosh 2v = \cosh^2 v + \sinh^2 v = 2\cosh^2 v - 1$$
$$= 2\sinh^2 v + 1$$
$$\sinh 2v = 2 \sinh v \cosh v$$

$$\operatorname{Cosh} v = \tfrac{1}{2}\{e^v + e^{-v}\}$$
$$= \frac{1}{2}\left\{1 + v + \frac{v^2}{\lfloor 2} + \frac{v^3}{\lfloor 3} + \text{etc.} + 1 - v + \frac{v^2}{\lfloor 2} - \frac{v^3}{\lfloor 3} + \text{etc.}\right\}$$
$$= 1 + \frac{v^2}{\lfloor 2} + \frac{v^4}{\lfloor 4} + \frac{v^6}{\lfloor 6} + \text{etc.}$$

$$\operatorname{Sinh} v = \tfrac{1}{2}\{e^v - e^{-v}\}$$
$$= \frac{1}{2}\left\{1 + v + \frac{v^2}{\lfloor 2} + \frac{v^3}{\lfloor 3} \ldots - \left(1 - v + \frac{v^2}{\lfloor 2} - \frac{v^3}{\lfloor 3} + \text{etc.}\right)\right\}$$
$$= v + \frac{v^3}{\lfloor 3} + \frac{v^5}{\lfloor 5} + \text{etc.}$$

## Inverse Hyperbolic Functions

Let $y = \sinh^{-1}\frac{x}{a}$ $\qquad \therefore \frac{x}{a} = \sinh y$

$$\therefore e^y - e^{-y} = \frac{2x}{a}$$

On solving for $e^y$ we have

$$e^{2y} - \frac{2x}{a}e^y + \left(\frac{x}{a}\right)^2 = 1 + \frac{x^2}{a^2}$$

$$\therefore e^y = \frac{x}{a} + \frac{\sqrt{a^2+x^2}}{a}$$

$$\therefore y = \log\frac{x+\sqrt{a^2+x^2}}{a} = \int_0^x \frac{dx}{\sqrt{a^2+x^2}}$$

$$\therefore \sinh^{-1}\frac{x}{a} = \int_0^x \frac{dx}{\sqrt{a^2+x^2}}$$

Similarly $\qquad \cosh^{-1}\frac{x}{a} = \int_a^x \frac{dx}{\sqrt{x^2-a^2}}$

## CHAPTER XXI

# SECOND MOMENT OR MOMENT OF INERTIA

If an element of area or mass be multiplied by its distance from any axis, the product is called the **first moment** of the element of area or mass about that axis. If the element of area or mass be multiplied by the square of its distance from the axis the product is called the **second moment** or **moment of inertia** of the element.

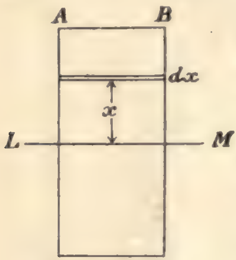

Fig. 73.

**To find the moment of inertia of a rectangle about a line, parallel to one of its sides, passing through its centre of area.**

The moment of inertia of an area about a line is the sum of each element of the area multiplied by the square of its distance from the line.

Let ABCD (Fig. 73) be a rectangle, LM a line through its centre parallel to AB

$$\text{Let AB} = b \text{ and BC} = d$$

Suppose the area split up into an indefinite number of strips, parallel to AB the breadth of each strip being $dx$ then an element of area is $bdx$ The moment of inertia of this element about LM is $x^2bdx$ where $x$ is its distance

from LM therefore the moment of inertia I of the rectangle about LM is

$$I = \int_{-\frac{d}{2}}^{+\frac{d}{2}} bx^2dx = b\left[\frac{x^3}{3}\right]_{-\frac{d}{2}}^{+\frac{d}{2}}$$

$$\therefore I = \frac{bd^3}{12}$$

$$I = Ak^2$$

where $k$ is the mean radius or radius of gyration.

$$\therefore k = \sqrt{\frac{I}{A}} \text{ for an area A}$$

$$\text{and } k = \sqrt{\frac{I}{M}} \text{ for a mass M}$$

**To find the moment of inertia of a circle about its centre, and also about a diameter.**

Let Fig. 74 represent a circle whose radius is $r$ and whose centre is at the origin.

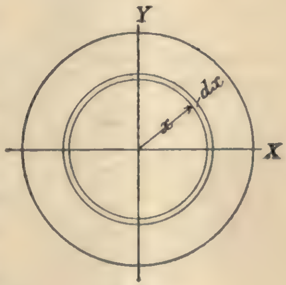

FIG. 74.

Suppose the area split up into an indefinite number of concentric strips, the breadth of each being $dx$ and let the radius of one of these strips be $x$ then its area is $2\pi x dx$ and its moment of inertia about the centre is $2\pi x^3 dx$ therefore the moment of inertia of the whole circle about its centre is

$$I = 2\pi\int_0^r x^3dx = 2\pi\left[\frac{x^4}{4}\right]_0^r$$

$$\therefore I = \frac{\pi r^4}{2} = \pi r^2 \times \frac{r^2}{2} = Ak^2$$

where A is its area, and $k$ is the *radius of gyration* which is $\frac{r}{\sqrt{2}}$

The moment of inertia of the circle about a diameter is half its moment of inertia about its centre, since

$$r^2 = x^2 + y^2$$

where $r$ is the distance of any element of area $a$ from the origin.

Now
$$ar^2 = ax^2 + ay^2$$
$$\therefore \Sigma ar^2 = \Sigma ax^2 + \Sigma ay^2$$

that is, $I_c$ about centre is equal to $I_x$ about axis of X plus $I_y$ about axis of Y but it is evident that

$$\Sigma ax^2 = \Sigma ay^2$$
$$\therefore I_x = \tfrac{1}{2} I_c$$
$$\therefore I_x = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

where $d$ is the diameter.

**To find the moment of inertia of a cylinder about its axis.**

The moment of inertia of a mass about any line is the sum of all such terms as each little element of mass multiplied by the square of its distance from the line. Let $l$ denote the length of the cylinder, $r$ the radius of its end, $\rho$ its density, therefore its mass is $\frac{\pi r^2 l\rho}{g}$ and since by the preceding example the radius of gyration is $\frac{r}{\sqrt{2}}$ it follows that

$$I = \frac{\pi r^2 l\rho}{g}\frac{r^2}{2} = \text{mass} \times \frac{r^2}{2}$$

**To find the moment of inertia of a sphere of mass, M, about its centre, and also about a diameter.**

Let $r$ denote the radius of the sphere, and suppose it divided up into an indefinite number of concentric shells of thickness $dx$ Let the radius of any one of these shells be $x$ its mass is therefore

$$4\pi m x^2 dx$$

where $m$ is the mass of unit volume and its moment of

inertia about the centre is

$$4\pi m x^4 dx$$

therefore the moment of inertia of the sphere about its centre is

$$I_c = 4\pi m\int_0^r x^4 dx = 4\pi m r^3 \times \frac{r^2}{5}$$

that is, $$I_c = \frac{4}{3}\pi m r^3 \times \frac{3r^2}{5} = \frac{3}{5}Mr^2$$

The moment of inertia $I_d$ of a sphere about a diameter is two-thirds of its moment of inertia about its centre.

For $$\Sigma mx^2 = \Sigma my^2 = \Sigma mz^2$$
and $$\Sigma mx^2 + \Sigma my^2 + \Sigma mz^2 = \Sigma mr^2$$
where $m$ is an element of mass; therefore

$$\Sigma mx^2 + \Sigma my^2 = \tfrac{2}{3}\Sigma mr^2$$

that is, the moment of inertia about the axis of $Z$ is $\frac{2}{3}$ of $I_c$

$$\therefore I_d = \tfrac{2}{3} \times \tfrac{3}{5}Mr^2 = \tfrac{2}{5}Mr^2$$

**Moment of inertia of a triangle about an axis through its vertex parallel to its base.**

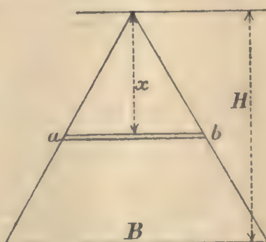

FIG. 75.

Let the base be B and the height H The length of the element $ab$ is

$$ab = \frac{Bx}{H}$$

$I_v$ of element about axis through the vertex is

$$\frac{B}{H}x^3dx$$

$$\therefore I_v = \frac{B}{H}\int_0^H x^3dx = \frac{B}{H}\left[\frac{x^4}{4}\right]_0^H = \frac{BH^3}{4} = \frac{BH}{2} \times \frac{H^2}{2}$$

$$\therefore I_v = \text{area} \times \frac{H^2}{2}$$

## Theorem of Parallel Axes

The moment of inertia of an area or mass about any axis is equal to its moment of inertia about a parallel axis through its centre of gravity plus the area or mass multiplied by the square of the distance between the axes.

In the figure let AB be an axis through the mass centre and let CD be a parallel axis at a distance $a$ from AB The moment of inertia of an element of mass $m$ about CD is

$$m(x+a)^2$$

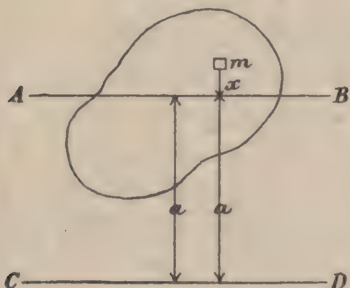


Fig. 76.

The moment of inertia I of the whole mass about CD is

$$I = \Sigma m(x+a)^2 = \Sigma mx^2 + \Sigma 2amx + \Sigma a^2m$$

But $\Sigma mx^2$ is the moment of inertia $I_0$ of the mass about AB and

$$\Sigma 2amx = 2a\Sigma mx = 0$$

also $\Sigma a^2m = a^2M$ where M is the whole mass;

therefore $$I = I_0 + a^2M \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(1) is also true for an area if we substitute A the area instead of M

$$\therefore I = I_0 + a^2A \quad . \quad . \quad . \quad . \quad . \quad (2)$$

**Moment of inertia $I_0$ of a triangle about an axis through its centre parallel to the base.**

By the theorem of parallel axes we have

$$I_v = I_0 + A\left(\frac{2H}{3}\right)^2$$

$$\therefore I_0 = I_v - A\frac{4}{9}H^2 = A\frac{H^2}{2} - \frac{4}{9}AH^2$$

$$\therefore I_0 = \frac{AH^2}{18}$$

**I of a triangle about its base.**

$$I_b = I_0 + A\left(\frac{H}{3}\right)^2 = \frac{AH^2}{18} + \frac{AH^2}{9} = \frac{AH^2}{6}$$

**Moment of inertia of a fly-wheel rim** the outer radius being R and the inner radius $r$

Let $b$ be the breadth of the rim measured parallel to the axis, $m$ the mass of unit volume.

Then an element of mass of radius $x$ is

$$2\pi bmxdx$$

and therefore

$$I = 2\pi bm\int_r^R x^3dx$$

$$= 2\pi mb\left[\frac{x^4}{4}\right]_r^R$$

$$= 2\pi bm\frac{R^4 - r^4}{4}$$

$$\therefore I = M\left(\frac{R^2 + r^2}{2}\right)$$

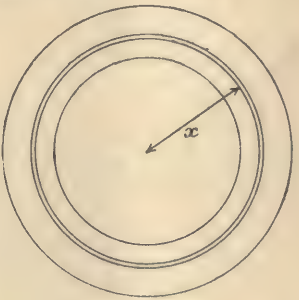


FIG. 77.

**I of a circular area about a tangent in its plane.** By the theorem of parallel axes, we have

$$I_t = I_d + AR^2 = \frac{\pi R^4}{4} + \pi R^4$$

$$\therefore I_t = \tfrac{5}{4}\pi R^4$$

**Moment of inertia of a circular area about a tangent perpendicular to its plane.**

$$I_t = I_c + AR^2$$
$$= \frac{\pi R^4}{2} + \pi R^4$$
$$\therefore I_t = \tfrac{3}{2}\pi R^4$$

**Moment of inertia of a cone about its axis.**

Let the height be H and the base B

Let unit volume have unit mass. The moment of inertia of an element of radius $y$ and thickness $dx$ is

$$\frac{\pi y^4}{2}dx$$

but $y = x\frac{R}{H}$ where H is the height and R is the radius of the base

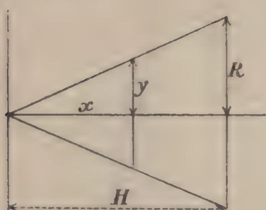


FIG. 78.

$$\therefore I = \frac{\pi R^4}{2H^4}\int_0^H x^4 dx$$
$$= \frac{\pi R^4}{2H^4}\left[\frac{x^5}{5}\right]_0^H$$
$$\therefore I = \frac{\pi R^4 H}{10}$$
$$= \frac{\pi R^2 H}{3} \times \frac{3R^2}{10}$$
$$= M\frac{3R^2}{10} \text{ where M is the mass.}$$

**Moment of inertia of a cone about an axis through its vertex parallel to its base.**

By the theorem of parallel axes the I of an element about the given axis is

$$\frac{\pi y^4}{4}dx + \pi y^2 x^2 dx$$

$$\therefore \mathrm{I}_r = \frac{\pi R^4}{4H^4}\int_0^H x^4 dx + \frac{\pi R^2}{H^2}\int_0^H x^4 dx$$

$$= \frac{\pi R^4}{4H^4}\left[\frac{x^5}{5}\right]_0^H + \frac{\pi R^2}{H^2}\left[\frac{x^5}{5}\right]_0^H$$

$$= \frac{\pi R^4 H}{20} + \frac{\pi R^2 H^3}{5}$$

$$= \frac{\pi R^2 H}{3}\left\{\frac{3R^2}{20} + \frac{3H^2}{5}\right\} = M\left\{\frac{3R^2}{20} + \frac{3}{5}H^2\right\}$$

where M is the mass.

**Moment of inertia of a long thin rod** about an axis through its mass centre perpendicular to its length.

$$d(\mathrm{I}) = mx^2 dx$$

where $m$ is the mass of unit length and $x$ is the distance of an element from the axis.

$$\therefore \mathrm{I} = m\int_{-\frac{l}{2}}^{+\frac{l}{2}} x^2 dx = m\left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{+\frac{l}{2}}$$

$$= \frac{ml^3}{12}$$

$$\therefore \mathrm{I} = M\frac{l^2}{12}$$

**Moment of inertia of a round bar or cylinder** about an axis through its centre of gravity perpendicular to its length.

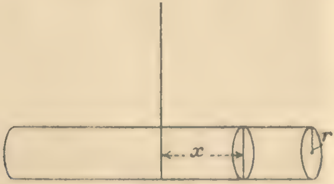


FIG. 79.

The moment of inertia of an element of thickness $dx$ at

a distance $x$ from the axis is, by the theorem of Parallel axes,

$$\frac{\pi r^4}{4}dx + \pi r^2 x^2 dx$$

$$\therefore \mathrm{I} = \frac{\pi r^4}{4}\int_{-\frac{l}{2}}^{+\frac{l}{2}} dx + \pi r^2 \int_{-\frac{l}{2}}^{+\frac{l}{2}} x^2 dx$$

$$= \frac{\pi r^4}{4} l + \pi r^2 \frac{l^3}{12}$$

$$\therefore \mathrm{I} = \mathrm{M}\left\{\frac{r^2}{4} + \frac{l^2}{12}\right\}$$

where M is the mass taking unit volume as unit mass.

## CHAPTER XXII

# SIMPLE HARMONIC MOTION, ETC.

LET a point P move round the circumference of a circle with a uniform velocity, and let D be the foot of the perpendicular on a diameter; then D moves with a *simple harmonic motion*. Let the displacement of D from its mean position at any instant be denoted by $x$ and let the angle DOP be $\theta$

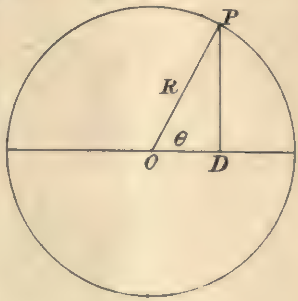


FIG. 80.

then $\qquad x = \mathrm{R}\cos\theta$

$$\frac{dx}{dt} = -\mathrm{R}\sin\theta\frac{d\theta}{dt} \quad (1)$$

but $\frac{dx}{dt}$ is the velocity of D

therefore (1) gives the velocity of D for any given value of $\theta$

and $\frac{d\theta}{dt}$ is the angular velocity of P

Suppose P to make 180 turns per minute

then $$\frac{d\theta}{dt} = \frac{180}{60} \times 2\pi \text{ radians per second.}$$

Let R = 2 ft, and $\theta$ = 30°, then the velocity of D is

$$2\sin 30°(6\pi) = 18{\cdot}8496 \text{ ft. per second.}$$

The negative sign in (1) indicates that $x$ diminishes as $t$ increases.

Differentiate (1) with regard to time, and we have

$$\frac{d^2x}{dt^2} = -\mathrm{R} \cos \theta \left(\frac{d\theta}{dt}\right)^2$$

Now $\frac{d^2x}{dt^2}$ is the acceleration of D along the diameter, and $\mathrm{R} \cos \theta = x$, the displacement of D

$$\therefore \text{Acceleration} = \text{Displacement} \left(\frac{d\theta}{dt}\right)^2$$

Let the periodic time be T seconds and therefore

$$\frac{2\pi}{\mathrm{T}} = \frac{d\theta}{dt}$$

$$\therefore \frac{4\pi^2}{\mathrm{T}^2} = \frac{\text{Acceleration}}{\text{Displacement}}$$

$$\therefore \mathrm{T} = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} \quad . \quad . \quad (2)$$

This is the law for any body moving with a *Simple Harmonic Motion.*

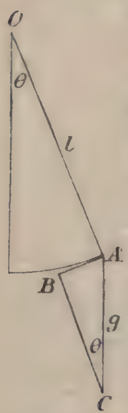

FIG. 81.

In the case of the simple pendulum let $l$ be the length in feet and let $\theta$ be the angular displacement, at any instant, from the vertical position, then the acceleration is $g \sin \theta$ which is represented by AB where AC represents $g$

$$\therefore \mathrm{T} = 2\pi\sqrt{\frac{l\theta}{g \sin \theta}}$$

If the amplitude of the vibration be small then $\frac{\theta}{\sin \theta}$ may be taken as unity and we have

$$\mathrm{T} = 2\pi\sqrt{\frac{l}{g}}$$

Formula (2) holds good if we substitute angular displacement and angular acceleration.

$$\therefore \mathrm{T} = 2\pi\sqrt{\frac{\text{Angular displacement}}{\text{Angular acceleration}}} \quad . \quad (3)$$

Take the case of a needle vibrating under the action of the torsion of a wire. Let the torque be $\mu$ poundals feet units when the angular displacement is one radian, therefore the torque is $\mu\theta$ when the angular displacement is $\theta$ radians. The angular acceleration A is given by

$$\mathrm{A} = \frac{\text{Torque}}{\mathrm{I}} = \frac{\mu\theta}{\mathrm{I}}$$

Substituting in (3) we have

$$\mathrm{T} = 2\pi\sqrt{\frac{\theta}{\frac{\mu\theta}{\mathrm{I}}}} = 2\pi\sqrt{\frac{\mathrm{I}}{\mu}}$$

where I is the moment of inertia of the needle about the axis of rotation.

*Example.*—Find the periodic time of vibration of a uniform needle, 2 feet long which weighs 6 lbs. when suspended horizontally by a wire which requires a torque of 30 poundals feet units to twist it through one radian.

Here $$\mathrm{T} = 2\pi\sqrt{\frac{\frac{Ml^2}{12}}{\mu}} = 2\pi\sqrt{\frac{6 \times \frac{1}{3}}{30}}.$$

$$= \frac{2\pi}{\sqrt{15}} \text{ seconds}$$

## The Compound Pendulum

Let O be the axis of suspension, G the centre of gravity of the pendulum and let OP be the length of the equivalent simple pendulum, and let $\theta$ be the angular displacement at any instant. Then the restoring torque is

$g\mathrm{M} \times \mathrm{OG} \sin\theta$ where M is the mass,

therefore the angular acceleration A is given by

$$A = \frac{gM \times OG \sin\theta}{I}$$

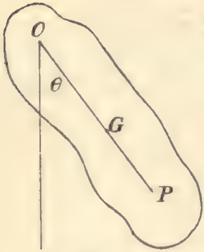

FIG. 82.

where I is the moment of inertia of the pendulum about the axis through O.

Substituting in (3) we have

$$T = 2\pi\sqrt{\frac{\theta}{\frac{gM \times OG \sin\theta}{I}}}$$

which reduces to

$$T = 2\pi\sqrt{\frac{K^2}{OG \times g}}$$

where K is the radius of gyration of the pendulum about the axis of suspension

$$K^2 = k^2 + OG^2$$

where $k$ is the radius of gyration about the axis through the centre of gravity parallel to the axis of suspension.

$$\therefore T = 2\pi\sqrt{\frac{k^2 + OG^2}{OG \times g}}$$

Now OP is the length of the equivalent simple pendulum, therefore

$$OP = \frac{k^2 + OG^2}{OG}$$

$$\therefore OP \times OG = k^2 + OG^2$$

$$\therefore OG(OP - OG) = k^2$$

that is, $$OG \times GP = k^2$$

This proves that *the centre of oscillation and centre of suspension are interchangeable.*

*Example.*—A uniform rod 6 feet long is suspended about

an axis perpendicular to its length at a distance of 2 feet from its centre, it is required to find the periodic time and the length of the equivalent simple pendulum

$$T = 2\pi\sqrt{\frac{k^2 + a^2}{ag}}$$

where $a$ is the distance from the centre of gravity of the rod to the axis of suspension

$$\therefore T = 2\pi\sqrt{\frac{\frac{6^2}{12} + 2^2}{2 \times 32 \cdot 2}} = 2 \cdot 07 \text{ sec. nearly}$$

Length of equivalent simple pendulum

$$= \frac{\frac{6^2}{12} + 2^2}{2} = 3 \cdot 5 \text{ feet}$$

**Periodic time of vibration of a column of water in a U tube**

Let the length of the column of water in the U tube be L feet, and suppose $x$ to be the height in feet of the water in one leg above its mean height, then the difference of level in the two legs will be $2x$ feet. Let $w$ be the weight of unit length of the column, then when the displacement is $x$ feet from the mean position, the restoring force is

$$2wxg$$

absolute units of force and therefore the acceleration is

$$\frac{2wxg}{wL}$$

$$T = 2\pi\sqrt{\frac{\text{Displacement}}{\text{Acceleration}}} = 2\pi\sqrt{\frac{x}{\frac{2xg}{L}}}$$

$$= 2\pi\sqrt{\frac{\frac{L}{2}}{g}}$$

That is, the periodic time will be the same as that of a simple pendulum the length of which is equal to half the length of the column of water. The periodic time of the explosions in the Humphrey pump depends upon the natural period of oscillation of the column of water in the pump and delivery pipe.

Periodic time of a thin hoop of diameter $d$ hanging on a peg.

Here
$$T = 2\pi\sqrt{\frac{k^2 + a^2}{ag}}$$

$$a = \frac{d}{2}, \text{ and } k = \frac{d}{2}$$

$$\therefore T = 2\pi\sqrt{\frac{\frac{d^2}{4} + \frac{d^2}{4}}{\frac{d}{2} \times g}}$$

$$= 2\pi\sqrt{\frac{d}{g}}$$

That is, the periodic time is the same as that of a simple pendulum the length of which is equal to the diameter of the hoop.

**Total pressure on an immersed area**

Let $w$ be the weight of unit volume of the liquid, A the area inclined at $\theta$ to the horizontal. Let $a$ be a very small portion of the area, and suppose the plane produced to cut the surface of the liquid in CD The pressure on an element of area $a$ is proportional to the vertical depth below the surface, therefore the pressure on $a$ at a distance $x$ from CD is

$$wax \sin \theta$$

The total pressure is $\quad \Sigma wax \sin \theta$

that is $\quad w \sin \theta \Sigma ax$

But by the principle of moments

$$w \sin \theta \Sigma ax = w \sin \theta A\bar{x}$$

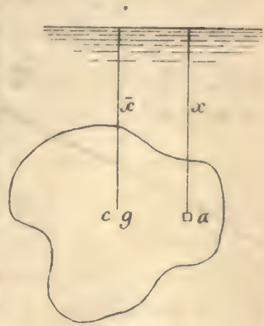

FIG. 83.

where $x$ is the distance of the centre of the area from CD, the vertical depth being $\bar{x} \sin \theta$ That is, the total pressure on an immersed area is obtained by multiplying together the area, the depth of the centre of gravity below the surface, and the weight of unit volume of the liquid.

If the vertical depth of the centre of area be $\bar{x}$, then the total pressure is

$$wA\bar{x}$$

**Centre of pressure** on an immersed area. Referring to the preceding figure, the pressure on an element of area $a$ is $wax \sin \theta$ and the moment of this pressure about the line CD is

$$wax^2 \sin \theta$$

The total moment is $\quad w \sin \theta \Sigma ax^2$

But this is equal to the moment of the total pressure, about CD

that is $\qquad \bar{X} w \sin \theta A\bar{x} = w \sin \theta \Sigma ax^2$

therefore $\qquad \bar{X} = \dfrac{w \sin \theta \Sigma ax^2}{w \sin \theta A\bar{x}} = \dfrac{\Sigma ax^2}{A\bar{x}}$

where $\bar{X}$ is the distance of the centre of pressure from the line CD

that is $\quad \bar{X} = \dfrac{\text{Moment of inertia of A about CD}}{\text{Distance of centre of area from CD} \times \text{A}}$

That is $\quad \bar{X} = \dfrac{AK^2}{A\bar{x}} = \dfrac{K^2}{\bar{x}} = \dfrac{k^2 + \bar{x}^2}{\bar{x}}$

where $k$ is the radius of gyration of the area about an axis through the centre of area parallel to CD

From the above result it can be deduced that

$$\bar{x}(\bar{X} - \bar{x}) = k^2$$

or the distance between the centre of pressure and the centre of area is

$$\bar{X} - \bar{x} = \frac{k^2}{\bar{x}} = d \text{ say}$$

$$\therefore d\bar{x} = k^2$$

*Example.*—Find the depth of the centre of pressure on a vertical circular area of radius 2 feet, whose centre is 6 feet below the surface.

$$\bar{x} = \frac{\frac{\pi r^4}{4} + \pi r^2 a^2}{\pi r^2 a} = \frac{\frac{r^2}{4} + a^2}{a}$$

where $r = 2$ and $a = 6$

$$\therefore \bar{x} = \tfrac{37}{6} = 6\tfrac{1}{6} \text{ feet} = 6 \text{ feet } 2 \text{ inches}$$

that is, the centre of pressure is 2 inches below the centre of the circle.

*Example.*—Find the total pressure and centre of pressure on a semicircular lock gate with the diameter in the surface; the diameter being 20 feet, $w = 62{\cdot}4$

$$\text{Total pressure} = \text{area} \times \text{depth of centre of area} \times w$$

$$= w\frac{\pi r^2}{2} \times \frac{4r}{3\pi} = \frac{2}{3}r^3 w$$

$$= \frac{2}{3} \times 10^3 \times 62{\cdot}4 \text{ lbs.}$$

Depth of centre of pressure,

$$\bar{X} = \frac{K^2}{\bar{x}} = \frac{\frac{r^2}{4}}{\frac{4r}{3\pi}} = \frac{3\pi r}{16}$$

$$= 5{\cdot}9 \text{ feet nearly}$$

**Centre of percussion** of a thin heavy rod hinged at one end.

Let $m$ be the mass of unit length of the rod hinged at O and let $l$ be the length of the rod.

Let $a$ be the acceleration of the rod at unit distance from the hinge, therefore the acceleration at a distance $x$ from the hinge is $ax$ The force required to produce this acceleration on a length $dx$ at a distance $x$ from the hinge is

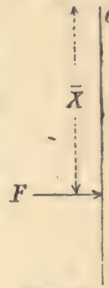

Fig. 84.

$$amxdx$$

and the total force F on the rod is

$$am\Sigma xdx$$

The moment of this force about the hinge is

$$\bar{X}am\int xdx$$

and this must be equal to the sum of the moments of the elements of force

therefore $$\bar{X}am\int_0^l xdx = am\int_0^l x^2dx \quad . \quad . \quad . \quad (1)$$

$$\therefore \bar{X} = \tfrac{2}{3}l$$

From this result we infer that if a long heavy uniform rod be struck a sudden blow at two-thirds of its length from one end it will, if free to move, begin to rotate about that end.

From (1) we deduce that

$$\bar{X} = \frac{I}{M\bar{x}} = \frac{K^2}{\bar{x}}$$

That is, the centre of percussion with respect to the hinge corresponds with the centre of oscillation of the rod regarded as a compound pendulum.

**To find the law connecting the tensions and angle of lapping of a belt on a pulley.**

Let O be the centre of the pulley, AOB the angle of lapping of the belt on the pulley $a$, $T_1$ the tension of the belt at A and T the tension at B and let $T > T_1$

Consider a very short portion of the belt between A and B Let the tension at one end of it be $t$ then the tension at the

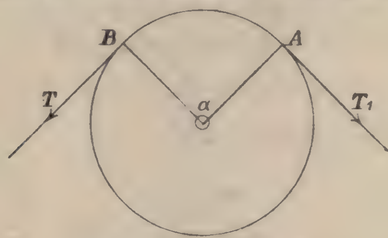


FIG. 85.

other end of it will be $t + dt$ Let R denote the reaction of the pulley. These three forces keep the small portion of the belt in equilibrium, therefore they are proportional to the three sides of a triangle. Draw the triangle of forces. We have the angle between the two forces $t$ and $t + dt$ equal

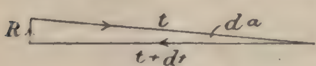


FIG. 86.

to the angle between the two radii at the extremities of the small portion under consideration. Denote this angle by $da$ and we have

$$R = tda$$

But the friction is $= \mu R$ where $\mu$ is the coefficient of friction between the belt and the pulley; that is, $dt = \mu R = \mu tda$

Therefore $$\frac{dt}{t} = \mu da$$

Integrating, we have

$$\log t = \mu a + k$$

Now $a = 0$ where $t = T_1$

$$\therefore \log T_1 = k$$

$$\therefore \log t = \mu a + \log T_1$$

that is,
$$\log \frac{t}{T_1} = \mu a$$

or
$$\frac{t}{T_1} = e^{\mu a}$$

Therefore
$$t = T_1 e^{\mu a}$$

This gives the tension at any point when $T_1$, $\mu$ and $a$ are given, when the belt is about to slip.

**To find the law for the thickness of a long pump-rod of uniform strength.**

Let $a$ denote its cross-section in square inches at any point, $f$ its safe stress; therefore its safe load

$$w = af \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

At a height $dh$ above this point the load is $w + acdh$ where $c$ is the weight of a cubic inch of the material of the rod; therefore, for uniform strength we must have

$$w + acdh = f(a + da) \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2) we have

$$acdh = fda$$

that is,
$$\frac{da}{a} = \frac{c}{f}dh$$

Integrating, we get

$$\log a = \frac{c}{f}h + C$$

Where $h = 0$ that is, at the lower end of the rod, let $a = a_0$

$$\therefore C = \log a_0$$

$$\therefore \log \frac{a}{a_0} = \frac{ch}{f}$$

$$\therefore a = a_0 e^{\frac{ch}{f}}$$

This law determines the cross-section at any height $h$ above its lower end.

Suppose $a_0 = \cdot 8$, $c = \cdot 28$, $h = 300$, and $f = 1000$

$$\therefore a = \cdot 8(2 \cdot 718)^{\cdot 084}$$
$$\log a = \bar{1} \cdot 9031 + \cdot 084 \times \cdot 4343$$
$$= \bar{1} \cdot 9369$$
$$\therefore a = \cdot 8702 \text{ sq. inch}$$

**To find the shape and deflection of a semi-girder of uniform section loaded at its extremity.**

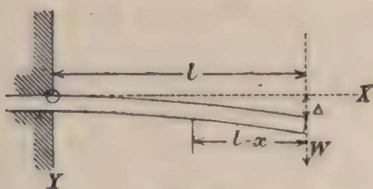


Fig. 87.

Let $l$ denote the length of the beam in inches, W the load in lbs. at its extremity, 0 the origin, 0X the axis of X and 0Y the axis of Y

At a distance $x$ from the end, the bending moment M produced by W is

$$M = (l - x)W$$

but the bending moment at a section is

$$M = \frac{EI}{R}$$

where E is the modulus of elasticity of the material of the beam, I the moment of inertia of the section about a hori-

zontal line through its centre of area, and R is the radius of curvature of the girder at the section.

$$\mathrm{R} = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

and since, in the case of beams, $\frac{dy}{dx}$ is so small that we may neglect it, therefore we have

$$\frac{1}{\mathrm{R}} = \frac{d^2y}{dx^2}$$

therefore $$\mathrm{EI}\frac{d^2y}{dx^2} = (l-x)\mathrm{W}$$

On integrating this differential equation once, we have

$$\mathrm{EI}\frac{dy}{dx} = \left(lx-\frac{x^2}{2}\right)\mathrm{W} + \text{constant}$$

Now $\frac{dy}{dx} = 0$ where $x = 0$ therefore the constant $= 0$ therefore we have

$$\mathrm{EI}\frac{dy}{dx} = \left(lx-\frac{x^2}{2}\right)\mathrm{W}$$

which gives the slope of the girder for a given value of $x$

Integrate again, and we have

$$\mathrm{EI}y = \left(\frac{lx^2}{2}-\frac{x^3}{6}\right)\mathrm{W}$$

No constant is added, since $y = 0$ when $x = 0$

$$\therefore y = \left(\frac{lx^2}{2}-\frac{x^3}{6}\right)\frac{\mathrm{W}}{\mathrm{EI}}$$

This gives the deflection $y$ for any given value of $x$ The

value of $y$ is greatest where $x = l$ therefore greatest deflection

$$\Delta = \frac{Wl^3}{3EI}$$

**To find the shape and deflection of a semi-girder of uniform section loaded uniformly with a load of *w* lb. per foot run.**

Let $l$ denote the length of the semi-girder in feet, the axis of X and Y as in the preceding example.

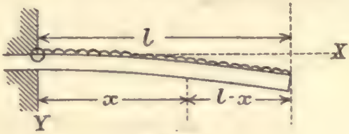


FIG. 88.

The bending moment at a distance $l-x$ from its extremity is

$$M = w(l-x)\frac{(l-x)}{2}$$

for the weight $w(l-x)$ acts as if it were collected at a distance $\frac{l-x}{2}$ from the section, therefore

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}(l-x)^2$$

Integrating, we have

$$EI\frac{dy}{dx} = \frac{w}{2}\left(l^2x - lx^2 + \frac{x^3}{3}\right)$$

No constant is added, since $\frac{dy}{dx} = 0$ where $x = 0$

This equation gives the slope $\frac{dy}{dx}$ of the girder for a given value of $x$ Integrate again, and we have

$$EIy = \frac{w}{2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right)$$

Here, again, no constant is added, since $y = 0$ where $x = 0$

This equation gives the deflection for a given value of $x$

The greatest deflection $\Delta$ is where $x = l$

$$\therefore \Delta = \frac{wl^4}{8EI} = \frac{Wl^3}{8EI}$$

where $W = wl$ the total load.

To get $\Delta$ in inches we must take $l$ in inches, and the unit of length must be one inch in calculating I

**To find the shape and deflection of a girder of uniform section loaded at its centre and supported at its ends.**

Let $l$ denote the distance between the two supports, W the load at the centre. Suppose one half of this girder to

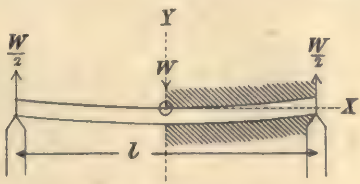


FIG. 89.

become embedded in a wall, then the shape of the beam will be unaltered, and the problem reduces to that given on page 232; but instead of W we have $\frac{W}{2}$ and instead of $l$ we have $\frac{l}{2}$ therefore the equation of the deflection curve is

$$EIy = \left\{\left(\frac{l}{2}\right)\frac{x^2}{2} - \frac{x^3}{6}\right\}\frac{W}{2}$$

where $y$ is the height above and $x$ the distance from the centre of the beam.

The greatest deflection is equal to the greatest value of $y$ that is, where $x = \frac{l}{2}$

$$\therefore \Delta = \frac{Wl^3}{48EI}$$

**To find the shape and the deflection of a girder of uniform section supported at its ends and loaded uniformly.**

Let $l$ denote the length of the girder in feet, $w$ the load per foot run. Taking the middle point 0 as origin, the

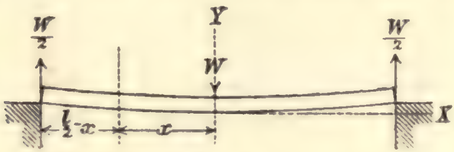


FIG. 90.

horizontal line through 0 as the axis of X and the vertical line through 0 as the axis of Y we have the bending moment at a distance $x$ from the origin,

$$M = \frac{wl}{2}\left(\frac{l}{2}-x\right)-\frac{w}{2}\left(\frac{l}{2}-x\right)^2 = \frac{w}{2}\left(\frac{l^2}{4}-x^2\right)$$

therefore $$EI\frac{d^2y}{dx^2} = \frac{w}{2}\left(\frac{l^2}{4}-x^2\right)$$

Integrating, we have

$$EI\frac{dy}{dx} = \frac{w}{2}\left(\frac{l^2x}{4}-\frac{x^3}{3}\right)$$

No constant is added, since $\frac{dy}{dx} = 0$ where $x = 0$

Integrate again; therefore

$$EIy = \frac{w}{2}\left(\frac{l^2x^2}{8}-\frac{x^4}{12}\right)$$

There is no constant added, since $y = 0$ when $x = 0$

The deflection is equal to the greatest value of $y$ that is, where $x = \frac{l}{2}$ therefore

$$\Delta = \frac{5wl^4}{384\mathrm{EI}} = \frac{5\mathrm{W}l^3}{384\mathrm{EI}} \text{ where } \mathrm{W} = wl$$

**To find the shape and deflection of a girder of uniform section fixed at its ends and loaded at its centre.**

Let $l$ denote the length of the girder in feet, W the load at its centre in lbs.

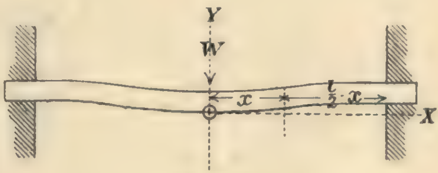


FIG. 91.

Taking the middle point 0 as origin and axis, as in the preceding example, we have the bending moment at a distance $x$ from the centre.

$$\mathrm{M} = \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right) - \mathrm{M}' \quad . \quad . \quad . \quad (a)$$

where M′ is the bending moment due to the stresses in the girder at its intersection with the wall. Therefore

$$\mathrm{EI}\frac{d^2y}{dx^2} = \frac{\mathrm{W}}{2}\left(\frac{l}{2} - x\right) - \mathrm{M}'$$

Integrating, we have

$$\mathrm{EI}\frac{dy}{dx} = \frac{\mathrm{W}}{2}\left(\frac{lx}{2} - \frac{x^2}{2}\right) - \mathrm{M}'x \quad . \quad . \quad (b)$$

There is no constant to be added, since $\frac{dy}{dx} = 0$ where $x = 0$

We have also $\frac{dy}{dx} = 0$ where $x = \frac{l}{2}$ and this gives

$M' = \frac{Wl}{8}$

The bending moment at the centre is got from ($a$) by substituting 0 for $x$ and $\frac{Wl}{8}$ for M′ therefore

$$M = \frac{Wl}{4} - \frac{Wl}{8} = \frac{Wl}{8}$$

Integrating ($b$), we have

$$EIy = \frac{W}{2}\left(\frac{lx^2}{4} - \frac{x^3}{6}\right) - \frac{Wlx^2}{16} \quad . \quad . \quad . \quad (c)$$

The constant is 0, since $y = 0$ where $x = 0$

The greatest deflection is got by substituting $\frac{l}{2}$ for $x$ in equation ($c$) therefore

$$\Delta = \frac{Wl^3}{192EI}$$

The points of inflection may be found from ($a$) by substituting 0 for M

that is $$0 = \frac{W}{2}\left(\frac{l}{2} - x\right) - \frac{Wl}{8}$$

Therefore $$x = \frac{l}{4}$$

that is, the middle segment is one-half the length of the girder.

**To find the shape and the deflection of a girder of uniform section fixed at its extremities and loaded uniformly.**

Let $l$ denote the length of the girder in feet, $w$ the load

per foot run, the axes as in the preceding example. The bending moment at a distance $x$ from the middle is

$$M = \frac{wl}{2}\left(\frac{l}{2}-x\right)-\frac{w}{2}\left(\frac{l}{2}-x\right)^2 - M' \quad . \quad . \quad (a)$$

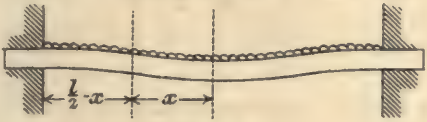

FIG. 92.

where M′ is the bending moment due to the stresses in the girder at its intersection with the wall; therefore

$$EI\frac{d^2y}{dx^2} = \frac{w}{2}\left(\frac{l^2}{4}-x^2\right) - M' \quad . \quad . \quad . \quad (b)$$

Integrating, we have

$$EI\frac{dy}{dx} = \frac{w}{2}\left(\frac{l^2x}{4}-\frac{x^3}{3}\right) - M'x \quad . \quad . \quad . \quad (c)$$

The constant being 0 since $\frac{dy}{dx} = 0$ where $x = 0$

Again, where $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$

therefore $M' = \frac{Wl}{12}$ where $W = wl$

Substituting in $(c)$ for M′ we have, on integrating,

$$EIy = \frac{w}{2}\left(\frac{l^2x^2}{8}-\frac{x^4}{12}\right)-\frac{Wlx^2}{24}$$

the constant being 0 since $y = 0$ where $x = 0$

The greatest deflection is got from this equation by substituting $\frac{l}{2}$ for $x$

Hence $\Delta = \frac{Wl^3}{384EI}$

The bending moment $M_c$ at the centre is got from ($a$) by substituting 0 for $x$ and $\frac{Wl}{12}$ for M′

that is $$M_c = \frac{wl}{2}\times\frac{l}{2}-\frac{w}{2}\times\left(\frac{l}{2}\right)^2-\frac{Wl}{12}$$

$$\therefore M_c = \frac{Wl}{24}$$

That is, the bending moment at the centre is half that at the intersection of the beam with the wall, or the beam is twice as ready to break at its intersection with the wall as at its centre.

The points of inflection may be found from ($a$) by making $M = 0$ Therefore

$$\frac{w}{2}\left(\frac{l^2}{4}-x^2\right) = \frac{wl^2}{12}$$

Hence $x = \pm\cdot288l$ therefore the middle segment $= \cdot576l$

**Reactions at the ends of a continuous beam over two equal spans.**

Let the load on AB be $w$ per foot run and that on BC

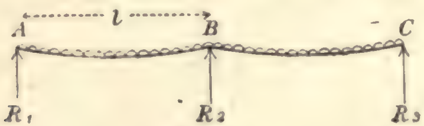


FIG. 93.

be $w_1$ per foot run, the spans being each $l$ feet. The bending moment at $x$ feet from the left-hand support is

$$R_1x-\frac{wx^2}{2} = -EI\frac{d^2y}{dx^2} \quad . \quad . \quad . \quad (1)$$

taking the straight line through ABC as the axis of X and the vertical through A drawn downwards as the axis of Y

In this case $\frac{d^2y}{dx^2}$ is negative because the curve is concave to the axis of X when $y$ is positive.

Integrating (1), we have

$$\mathrm{EI}\frac{dy}{dx} = \frac{wx^3}{6} - \mathrm{R}_1\frac{x^2}{2} + \text{constant} \quad . \quad . \quad (2)$$

Let $\frac{dy}{dx} = \tan\theta$ where $x = l$

$$\therefore \mathrm{EI}\tan\theta = \frac{wl^3}{6} - \mathrm{R}_1\frac{l^2}{2} + \text{constant} \quad . \quad (3)$$

Subtracting (3) from (2), we have

$$\mathrm{EI}\left\{\frac{dy}{dx} - \tan\theta\right\} = \frac{w}{6}\{x^3 - l^3\} - \frac{\mathrm{R}_1}{2}\left(x^2 - l^2\right)$$

Integrating again, we have

$$\mathrm{EI}(y - x\tan\theta) = \frac{w}{6}\left\{\frac{x^4}{4} - l^3x\right\} - \frac{\mathrm{R}_1}{2}\left\{\frac{x^3}{3} - l^2x\right\}$$

No constant is added since $y = 0$ where $x = 0$

At B $y = 0$ where $x = l$

$$\therefore -\mathrm{EI}l\tan\theta = \frac{w}{6}\left\{\frac{l^4}{4} - l^4\right\} - \frac{\mathrm{R}_1}{2}\left\{\frac{l^3}{3} - l^3\right\}$$

$$\therefore \tan\theta = \frac{l^2}{24\mathrm{EI}}\{3wl - 8\mathrm{R}_1\}$$

Similarly $\tan\theta = \frac{l^2}{24\mathrm{EI}}\{8\mathrm{R}_3 - 3w_1l\}$

Since $\tan\theta$ is negative reckoned from the other end of the beam,

$$\therefore 8\mathrm{R}_3 - 3w_1l = 3wl - 8\mathrm{R}_1$$

$$\therefore \mathrm{R}_1 + \mathrm{R}_3 = \tfrac{3}{8}(wl + w_1l) \quad . \quad . \quad (4)$$

Taking moments about B

we have $$\mathrm{R}_1l - \frac{wl^2}{2} = \mathrm{R}_3l - \frac{w_1l^2}{2} \quad . \quad . \quad . \quad . \quad (5)$$

Also $$\mathrm{R}_1 + \mathrm{R}_2 + \mathrm{R}_3 = wl + w_1l \quad . \quad . \quad . \quad (6)$$

From (4), (5) and (6) we obtain

$$R_1 = \frac{(7w - w_1)l}{16}$$

$$R_2 = \tfrac{5}{8}(w + w_1)l$$

and
$$R_3 = \frac{(7w_1 - w)l}{16}$$

If $w = w_1$ $R_1 = \frac{3}{8}wl = R_3$

and $R_2 = \frac{5}{4}wl$

**Simpson's Rule** for finding an area approximately.

Let $y_0$ $y_1$ $y_2$ be three consecutive ordinates of a curve, and let $x_0$ be the common distance between the ordinates.

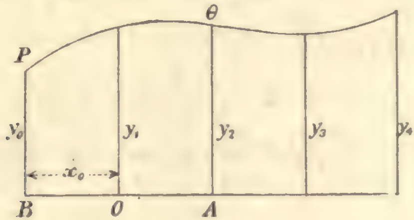


FIG. 94.

Assuming the curve PQ to be parabolic, its equation may be represented by

$$y = a + \beta x + \gamma x^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Taking 0 as the origin, we have

$$y_1 = a \quad \text{where } x = 0$$

and
$$y_0 = a - \beta x_0 + \gamma x_0^2 \quad . \quad . \quad . \quad . \quad (2)$$

$$y_2 = a + \beta x_0 + \gamma x_0^2 \quad . \quad . \quad . \quad . \quad (3)$$

Adding (2) and (3) we get

$$y_0 + y_2 = 2(a + \gamma x_0^2)$$

The area enclosed by the axis of X the ordinates $y_0$ and $y_2$ and the curve PQ is

$$\int_{-x_0}^{+x_0}(a+\beta x+\gamma x^2)dx$$

$$= \left[ax+\frac{\beta x^2}{2}+\frac{\gamma x^3}{3}\right]_{-x_0}^{+x_0}$$

$$\therefore \text{ area BPQA } = \left\{2ax_0+\frac{2\gamma x_0{}^3}{3}\right\}$$

$$= \frac{x_0}{3}\{6a+2\gamma x_0{}^2\}$$

$$= \frac{x_0}{3}\{4a+2a+2\gamma x_0{}^2\}$$

$$= \frac{x_0}{3}\{4y_1+y_0+y_2\} \quad . \; . \; . \quad (4)$$

Similarly the area included between the ordinates $y_2$ and $y_4$ the curve and the axis of X is

$$\frac{x_0}{3}\{4y_3+y_2+y_4\} \quad . \; . \; . \; . \; . \quad (5)$$

and that between $y_4$ and $y_6$ the axis of X and the curve is

$$\frac{x_0}{3}\{4y_5+y_4+y_6\} \quad . \; . \; . \; . \; . \quad (6)$$

On adding (4), (5) and (6) we have

$$\text{Area} = \frac{x_0}{3}\{(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\}$$

This result may be extended to any number of odd ordinates, that is, the number of spaces must be even.

Simpson's Rule may be extended to find a volume approximately if we substitute, instead of ordinates, areas of cross-sections at equal distances $x_0$ apart. Let the end areas be denoted by $A_0$ $A_{10}$ and the intermediate areas by $A_1$ $A_2$ $A_3$ etc., then the volume

$$V = \frac{x_0}{3}[(A_0+A_{10})+4(A_1+A_3+A_5+A_7+A_9) + 2(A_2+A_4+A_6+A_8)]$$

where there are 10 sections.

**Drop in temperature of steam flowing along a pipe.** Superheated steam is conveyed by a pipe 100 feet long from a superheater to an engine, the diameter of the pipe being 6 inches. Assuming that the heat lost over the surface of the pipe per square foot per hour obeys the law

$$h = \cdot 26(t - 68^\circ)$$

where $t$ is the temperature of the steam at a distance $x$ feet from the superheater, the air temperature being 68° Fahr. The weight of steam flowing per minute being 60 lbs., find the temperature at the stop-valve, and anywhere along the pipe, the temperature at the superheater being 500° Fahr.

Let W lbs. of steam flow along the pipe per hour the specific heat being $\sigma$. Let the temperature at $x$ feet from the superheater be $t$ and at a distance $x + dx$ let the temperature be $t - dt$ therefore the heat given out by the steam is

$$\mathrm{W}\sigma dt = -\cdot 26(t - 68^\circ)\pi \mathrm{D} dx$$

$$\therefore \frac{dt}{t - 68^\circ} = -\frac{\cdot 26\pi \mathrm{D} dx}{\mathrm{W}\sigma}$$

Integrating, we have

$$\log_e (t - 68^\circ) = -\frac{\cdot 26\pi \mathrm{D} x}{\mathrm{W}\sigma} + \text{constant}$$

$$t = 500^\circ, \text{ where } x = 0$$

$$\therefore \log (500 - 68) = \text{constant}$$

$$\therefore \log_e (t - 68^\circ) = -\frac{\cdot 26\pi \mathrm{D} x}{\mathrm{W}\sigma} + \log_e 432 \quad (1)$$

For steam $\sigma = \cdot 48$ In this example $\mathrm{W} = 3600$ $\mathrm{D} = \cdot 5$ foot, and $x = 100$ at stop-valve.

$$\therefore \log_e (t - 68^\circ) = -\frac{\cdot 26 \times 3\cdot 1416 \times \cdot 5 \times 100}{3600 \times \cdot 48} + \log_e 432$$

$$\log_{10} (t - 68^\circ) = 2\cdot 6252$$

$$\therefore t - 68^\circ = 421\cdot 9^\circ \text{ Fahr.}$$

$$\therefore t = 489\cdot 9^\circ \text{ Fahr.}$$

The drop in temperature is therefore 10·1° Fahr. (1) gives

the temperature at any given distance $x$ feet from the superheater.

The subnormal to a plane curve is given by

$$y\frac{dy}{dx}$$

If the subnormal be constant, then

$$\begin{aligned} y\frac{dy}{dx} &= c \text{ a constant} \\ \therefore\ y\,dy &= c\,dx \\ \therefore\ y^2 &= 2cx \end{aligned}$$

This is the equation to a parabola.

A cylindrical vessel 2 feet in diameter is filled with water to a depth of 2 feet, and is set rotating about its vertical axis. It is required to find the speed when a point on the bottom becomes bare.

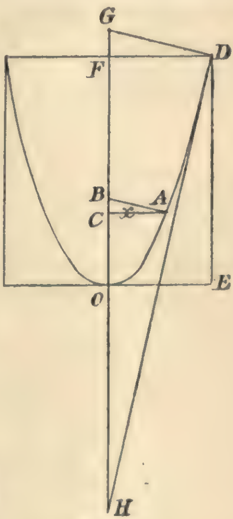

FIG. 95.

Let a particle of water be at a distance $x$ from the axis of rotation. There are three forces acting on it, namely the centrifugal force, its weight and the liquid pressure. These forces are in equilibrium and are proportional to the sides of the triangle ABC

$$\therefore \frac{\mathrm{BC}}{x} = \frac{w}{\frac{wv^2}{gx}} = \frac{gx}{v^2}$$

$$\therefore \mathrm{BC} = g\frac{x^2}{v^2} = \frac{g}{\omega^2}$$

where $\omega$ is the angular velocity in radians per second therefore BC is constant. This proves that the surface is a paraboloid of revolution, since the subnormal BC is constant.

The volume of a paraboloid of revolution is equal to half

the volume of the circumscribing cylinder, and therefore the water must rise up the sides to a distance of 4 feet when a point on the bottom becomes bare.

$$\therefore \text{DE} = 4 \text{ feet}$$

Now $\text{FH} = 2\text{OF} = 8$ feet and $\text{FD} = 1$ foot

and $$\text{GF} : \text{FD} :: \text{FD} : \text{FH}$$

$$\therefore \text{GF} = \tfrac{1}{8} = \text{BC}$$

$$\therefore \frac{1}{8} = \frac{g}{\omega^2}$$

$$\therefore \omega = \sqrt{8 \times 32} = 16 \text{ radians per sec.}$$

$$\therefore \text{revs. per sec.} = \frac{16}{2\pi} = 2{\cdot}54$$

**Average watts in an electric circuit.**

Let the E.M.F. obey the law $E = E_0 \sin pt$, and first suppose the current in phase with E.M.F. $I = I_0 \sin pt$ T being the periodic time.

$$\text{Average watts} = \frac{1}{T}\int_0^T E_0 I_0 \sin^2 pt\,dt$$

$$= \frac{E_0 I_0}{T}\left[\frac{t}{2} - \frac{\sin 2pt}{2p}\right]_0^T = \frac{E_0 I_0}{2}$$

That is, the average watts is one-half the product of the maximum E.M.F. and maximum value of the current.

Again, suppose the current to lag behind the E.M.F. by an angle $\theta$ then

$$I = I_0 \sin (pt - \theta)$$

the average watts will be

$$\frac{I_0 E_0}{T}\int_0^T \sin pt \sin (pt - \theta) dt$$

$$= \frac{I_0 E_0}{2T}\int_0^{T} \{\cos \theta - \cos (2pt - \theta)\} dt$$

$$= \frac{I_0 E_0}{2T} \cos \theta \Big[t\Big]_0^T - \frac{I_0 E_0}{2T} \times \frac{1}{2p}\Big[\sin (2pt - \theta)\Big]_0^T$$

$$= \frac{I_0 E_0}{2} \cos \theta$$

since $$\int_0^T \sin (pt - \theta) dt = 0$$

**Time of emptying a vessel under a varying head.**

Let the cross-section of the vessel be constant and equal to A square feet, and let H be the head in feet. Let $a$ be the area of the orifice, through which the water flows, in square feet.

Let the surface be lowered $dx$ feet in time $dt$ then the quantity of water which flows out in time $dt$ is

$$Adx = kavdt$$

where $k$ is the coefficient of discharge, and $v$ is the velocity due to the head $x$ feet

$$v = \sqrt{2g}x^{\frac{1}{2}}$$

$$\therefore Adx = ka\sqrt{2g}x^{\frac{1}{2}}dt$$

$$\therefore dt = \frac{Ax^{-\frac{1}{2}}dx}{ka\sqrt{2g}}$$

$$\therefore t = \frac{A}{ka\sqrt{2g}}\int_0^H x^{-\frac{1}{2}}dx = \frac{A}{ka\sqrt{2g}}\Big[2x^{\frac{1}{2}}\Big]_0^H$$

$$t = \frac{2A\sqrt{H}}{ka\sqrt{2g}} \text{ seconds}$$

Let the vessel be in the shape of an inverted cone, then the cross-section when the depth of water is $x$ feet is $A.\frac{x^2}{H^2}$ where A is the area of the base of the cone.

$$\therefore \frac{A}{H^2}x^2dx = ka\sqrt{2g}x^{\frac{1}{2}}dt$$

$$dt = \frac{Ax^{\frac{3}{2}}dx}{H^2ka\sqrt{2g}}$$

$$\therefore t = \frac{A}{H^2ka\sqrt{2g}}\int_0^H x^{\frac{3}{2}}dx$$

$$= \frac{A}{H^2ka\sqrt{2g}}\Big[\frac{2H^{\frac{5}{2}}}{5}\Big]_0^H$$

$$\therefore t = \frac{2AH^{\frac{1}{2}}}{5ka\sqrt{2g}} \text{ seconds}$$

Given A = 100 square feet H = 25 feet $a$ = ·25 square foot, $k$ = ·62, find $t$ when A is constant.

$$t = \frac{2 \times 100 \times 5}{\cdot 62 \times \cdot 25 \times 8} \text{ seconds} = 13{\cdot}44 \text{ minutes}$$

Given A = 100 square feet H = 25 feet, vessel in the form of an inverted cone, $a$ = ·25 square foot, find $t$.

$$t = \frac{2 \times 100 \times 5}{5 \times \cdot 62 \times \cdot 25 \times 8} = 161{\cdot}3 \text{ sec.} = 2{\cdot}688 \text{ minutes}$$

Two vessels of equal cross-section and height are connected by a short pipe. One vessel is filled with water and the other is empty; if the short pipe be opened, find the time taken until the water will be level in both vessels, assuming the coefficient of discharge to be ·62

Let the head in one vessel be $\frac{H}{2} + x$ then the head in the other vessel is $\frac{H}{2} - x$ The effective head will be $2x$ feet

$$\therefore A\,dx = \cdot 62 a\sqrt{2g}(2x)^{\frac{1}{2}}dt$$

$$\therefore dt = \frac{Ax^{-\frac{1}{2}}dx}{\cdot 62 a\sqrt{2g}\sqrt{2}}$$

$$\therefore t = \frac{A}{\cdot 62 a\sqrt{2g}\sqrt{2}}\int_0^{\frac{H}{2}} x^{-\frac{1}{2}}dx = \frac{2A}{\cdot 62 a\sqrt{2g}\sqrt{2}}\Big[x^{\frac{1}{2}}\Big]_0^{\frac{H}{2}}$$

$$= \frac{2A\left(\frac{H}{2}\right)^{\frac{1}{2}}}{\cdot 62 a\sqrt{2g}\sqrt{2}} = \frac{AH^{\frac{1}{2}}}{\cdot 62 a\sqrt{2g}} \text{ seconds}$$

Let A = 60 square feet H = 16 feet, $a$ = ·5 square foot

$$t = \frac{60 \times 4}{\cdot 62 \times \cdot 5 \times 8} \text{ secs.} = 96{\cdot}77 \text{ secs.} = 1{\cdot}613 \text{ minutes}$$

**Theoretical flow of water through a rectangular notch.**

Let B be the breadth of the notch in feet and H the depth in feet of the sill below the still water level. It is assumed that the velocity of flow in any section is that due to the height $x$ from the section to the still water level, therefore the velocity at depth $x$ feet is

$$\sqrt{2gx}$$

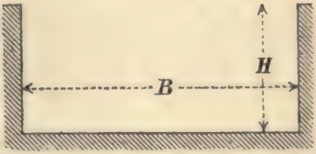


FIG. 96.

Denoting an element of quantity which flows through per second by $dQ$

we have

$$dQ = \sqrt{2gx}\,B\,dx$$

$$\therefore Q = B\sqrt{2g}\int_0^H x^{\frac{1}{2}}dx$$

$$= \tfrac{2}{3}B\sqrt{2g}\,H^{\frac{3}{2}}$$

$$= \tfrac{2}{3}BH\sqrt{2gH} \text{ cubic feet per second}$$

That is Q is equal to two-thirds of the area of the cross-section multiplied by the velocity due to the head H The actual flow is obtained by multiplying the theoretical flow by the coefficient of discharge which varies with the ratio of B to H

**Theoretical flow through a V-shaped weir gauge.**

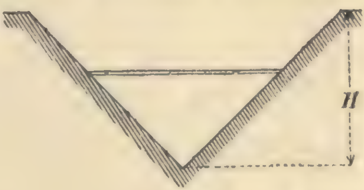


FIG. 97.

An element of area at a depth $x$ below the surface is

$$2(H - x)dx$$

and the velocity is $\sqrt{2gx}$

$$\therefore dQ = 2\sqrt{2gx}(H - x)dx$$

$$\therefore Q = 2\sqrt{2g}\int_0^H (Hx^{\frac{1}{2}} - x^{\frac{3}{2}})dx$$

$$= \tfrac{8}{15}\sqrt{2g}H^{\frac{5}{2}}$$

Actual flow is $\dfrac{8c}{15}\sqrt{2g}H^{\frac{5}{2}}$

where $c$ is the coefficient of discharge.

**Work done in lifting a body from the earth's centre to an infinite height against gravity.**

Let the body weigh W lbs. at the earth's surface, then at the centre of the earth its weight is zero, and the weight is directly proportional to the distance from the centre until the surface of the earth is reached, and above the surface the weight varies inversely as the square of the distance from the centre of the earth.

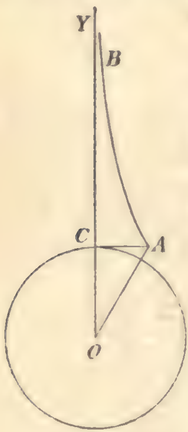


FIG. 98.

The distance of the curve OAB from the vertical OY shows how the weight varies. AC = W

Work done in bringing weight to the surface is

$$\frac{W}{2} \times R$$

where R is the radius of the earth.

Work done in lifting the body to an infinite height from the surface is

$$WR^2\int_R^\infty \frac{dx}{x^2}$$

$$= WR^2\Big[-\frac{1}{x}\Big]_R^\infty = WR$$

**Work done by a perfect gas expanding isothermally.**

The law of the expansion is

$$PV = c \text{ a constant}$$

where P is the pressure in pounds per square foot and V is the volume in cubic feet. Let the volume expand from $V_1$ to $V_2$ then

$$d\,(\text{work}) = P dV = c\frac{dV}{V}$$

$$\therefore \text{work} = c\int_{V_1}^{V_2}\frac{dV}{V} = c \log \frac{V_2}{V_1}$$

Suppose the initial pressure to be 14,400 lbs. per square foot and the volume 2 cubic feet, and let it expand to 10 cubic feet, then

$$\begin{aligned} \text{Work} &= P_1V_1 \log \frac{V_2}{V_1} \\ &= 14{,}400 \times 2 \log_e 5 \\ &= 14{,}400 \times 2 \times 2{\cdot}302 \times {\cdot}6990 \text{ ft.-lbs.} \end{aligned}$$

**Work done in compressing a gas adiabatically from $V_2$ to $V_1$**

The law of the expansion or compression is

$$PV^\gamma = c_1 \text{ constant}$$

$$d\,(\text{work}) = P dV = c.\frac{dV}{V^\gamma}$$

$$\begin{aligned} \therefore \text{work} &= c\int_{V_1}^{V_2}\frac{dV}{V^\gamma} \\ &= c\int_{V_1}^{V_2} V^{-\gamma}dV = c\left[\frac{V^{1-\gamma}}{1-\gamma}\right]_{V_1}^{V_2} \\ &= \frac{c}{\gamma-1}\left[V_1{}^{1-\gamma} - V_2{}^{1-\gamma}\right] \\ &= \frac{cV_1{}^{1-\gamma} - cV_2{}^{1-\gamma}}{\gamma-1} = \frac{P_1V_1 - P_2V_2}{\gamma-1} \end{aligned}$$

If the pressure be in lbs. per square inch and the volume in cubic feet

$$\text{work} = \frac{144(P_1V_1 - P_2V_2)}{\gamma - 1} \text{ ft.-lbs.}$$

To deliver the gas requires work

$$144P_1V_1 \text{ ft.-lbs. extra.}$$

**Extension of a long tapering tie-rod when loaded.** Let the diameter at the upper end be 4 inches and at the lower end 1 inch. Let the length of the rod be 30 feet, and suppose the load to be 10 tons.

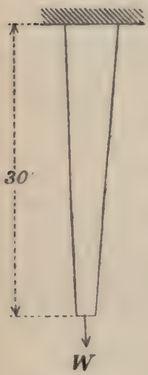

FIG. 99.

The modulus of elasticity being 13,000 tons.

The cross-section at a distance of $x$ feet from the lower end is

$$\frac{4\pi(x+10)^2}{1600} \text{ square inches}$$

The extension in a length $dx$ is

$$\frac{Wdx}{AE} = \frac{10dx}{\frac{4\pi(x+10)^2}{1600} \times 13000}$$

The total extension is therefore

$$= \int_0^{30} \frac{16dx}{52\pi(x+10)^2}$$

$$= \frac{4}{13\pi}\int\frac{dx}{(x+10)^2} = \frac{4}{13\pi}\left[-\frac{1}{x+10}\right]_0^{30}$$

$$= \frac{4}{13\pi} \times \left[\frac{1}{10} - \frac{1}{40}\right]$$

$$= \frac{3}{130\pi} \text{ inch}$$

**To investigate the rule for the strength of a thick cylindric pipe.**

Let Fig. 100 denote a section of a thick cylindric pipe

whose internal and external radii are $r$ and R respectively, and let the internal pressure be $p'$ lbs. per square inch and the external pressure $q$ lbs. per square inch, which may be neglected, the safe tensile stress of the material $f'$

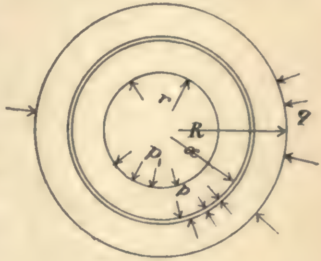

FIG. 100.

Consider a thin cylindric coaxial shell of the pipe of radius $x$ and thickness $dx$ and let the internal pressure on this shell be $p$ lbs. per square inch, and the external pressure $p + dp$ lbs. per square inch.

The former pressure tends to burst the shell and the latter tends to crush it. Denoting the tensile stress in this shell by $f$ we have

$$2px - 2(p + dp)(x + dx) = 2fdx$$

and neglecting the differential of the second order, we have

$$(f + p)dx + xdp = 0 \quad . \quad . \quad . \quad (a)$$

Again, assuming that a plane section, before the pressure acts, remains a plane section under pressure, we have

$$p - f = c \text{ a constant}$$

therefore $f = p - c$ Substituting for $f$ in equation $(a)$ we get

$$(2p - c)dx + xdp = 0$$

that is,
$$\frac{dx}{x} + \frac{dp}{2p - c} = 0$$

Integrating, we have

$$\log x + \tfrac{1}{2} \log (2p - c) = \text{constant}$$

therefore
$$2 \log x + \log (2p - c) = \text{constant}$$

Hence $x^2(2p - c) = 2\text{A}$ where A is a constant;

$$\therefore p = \frac{\text{A}}{x^2} + \frac{c}{2}$$

This law shows how the pressure diminishes as $x$ increases.

$p = p'$ where $x = r$, and neglecting the external pressure, $p = 0$ where $x = R$ therefore

$$0 = \frac{A}{R^2} + \frac{c}{2} \text{ and } p' = \frac{A}{r^2} + \frac{c}{2}$$

Hence $A = \dfrac{p'R^2r^2}{R^2 - r^2}$ and $\dfrac{c}{2} = -\dfrac{p'r^2}{R^2 - r^2}$

But $f' = p' - c$ therefore

$$f' = p' + \frac{2p'r^2}{R^2 - r^2}$$

Therefore $$\frac{f'}{p'} = \frac{R^2 + r^2}{R^2 - r^2}$$

or thus $$\frac{R}{r} = \sqrt{\frac{f' + p'}{f' - p'}}$$

# CHAPTER XXIII

## DIFFERENTIAL EQUATIONS

**A differential equation** is one which involves differentials or differential coefficients. It may or may not contain the primitive variables from which the differential coefficients are derived.

There are two classes of differential equations.

1st. *Differential equations in which all the differential coefficients involved have reference to a single independent variable.*

For example:

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0$$

is a differential equation in which $x$ is the independent and $y$ the dependent variable.

2nd. *Differential equations involving partial differential coefficients which indicate the presence of two or more independent variables.*

For example:

$$du = \left(\frac{du}{dx}\right)dx + \left(\frac{du}{dy}\right)dy$$

is a partial differential equation, having $u$ for its dependent and $x$ and $y$ for its independent variables. A differential equation is said to be of the first, second, third . . . or $n$th order, according as it contains the first, second, third . . . or $n$th differential coefficient.

$$\frac{dy}{dx} + Ay = B$$

is of the first order.

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

is of the second order.

$$\frac{d^ny}{dx^n} + a\frac{d^{n-1}y}{dx^{n-1}} \dots ny = 0$$

is of the $n$th order.

A differential equation is said to be of the first, second, third . . . or $n$th degree, according as it contains differential coefficients of the first, second, third . . . or $n$th degree.

$$\frac{dy}{dx} + ay = b$$

is of the first degree.

$$\left(\frac{dy}{dx}\right)^2 + a\frac{dy}{dx} + by = 0$$

is of the second degree.

$$\left(\frac{dy}{dx}\right)^n + a\left(\frac{dy}{dx}\right)^{n-1} \dots ny = 0$$

is of the $n$th degree.

A differential equation is said to be linear if it admits of being expressed in the form

$$\frac{d^ny}{dx^n} + a\frac{d^{n-1}y}{dx^{n-1}} + b\frac{d^{n-2}y}{dx^{n-2}} + \dots py = q$$

in which $a$, $b$, . . . $p$, and $q$ are either constants or functions of the independent variable $x$

If $a$ $b$ . . . $p$ be constants, it is said to be a linear differential equation with constant coefficients; for example:

$$\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0$$

is a linear differential equation with constant coefficients.

If $a\ b\ \ldots\ p$ are functions of $x$ it is said to be a differential equation with variable coefficients; for example:

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2y = 0$$

is a differential equation with variable coefficients.

**Solution of differential equations in which the variables are separable.**

The general form of differential equations of the first order and degree is

$$\mathrm{A}dx+\mathrm{B}dy = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

A and B being functions of the variables $x$ and $y$ In this form either $x$ or $y$ may be regarded as the independent variable. Equation (1) can always be solved if the variable admit of being separated and expressed in the form

$$\mathrm{X}dx+\mathrm{Y}dy = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where X and Y are functions of $x$ and $y$ respectively. The solution of (2) is expressed thus

$$\int\mathrm{X}dx+\int\mathrm{Y}dy = k$$

where $k$ is an arbitrary constant.

*Examples on the above form.*

$$\text{Solution of } (x^2-3x)dx+(y^2-2y)dy = 0$$

$$\int(x^2-3x)dx+\int(y^2-2y)dy = k$$

$$\therefore \frac{x^3}{3}-\frac{3x^2}{2}+\frac{y^3}{3}-y^2 = k$$

The solution of a differential equation is finding a relation between the variables free from differentials or differential coefficients.

Solution of $y dx + x^2 dy = 0$

In this example the variables may be separated by dividing by $yx^2$ thus

$$\frac{dx}{x^2} + \frac{dy}{y} = 0$$

$$\therefore \int\frac{dx}{x^2} + \int\frac{dy}{y} = k$$

$$\therefore -\frac{1}{x} + \log y = k$$

Solution of $\sqrt{a^2 + y^2}dx - \sqrt{b^2 - x^2}dy = 0$

On dividing by $\sqrt{(a^2 + y^2)(b^2 - x^2)}$ we have

$$\frac{dx}{\sqrt{b^2 - x^2}} - \frac{dy}{\sqrt{a^2 + y^2}} = 0$$

$$\therefore \int\frac{dx}{\sqrt{b^2 - x^2}} - \int\frac{dy}{\sqrt{a^2 + y^2}} = k$$

$$\therefore \sin^{-1}\left(\frac{x}{b}\right) - \log\left(y + \sqrt{y^2 + a^2}\right) = k$$

Solution of $\dfrac{x^3 + 1}{y + 1} = xy\dfrac{dy}{dx}$

Here $(x^3 + 1)dx = xy(y + 1)dy$ and on dividing by $x$ we have

$$\left(x^2 + \frac{1}{x}\right)dx = (y^2 + y)dy$$

$$\therefore \frac{x^3}{3} + \log x = \frac{y^3}{3} + \frac{y^2}{2} + k$$

**Solution of linear equations of the first order.**

A linear differential equation of the first order may be written in the form

$$\frac{dy}{dx} + Py = Q \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where P and Q are functions of $x$ The solution of this equation is important as it has several practical applications in engineering problems.

*Solution.*—Multiply both sides of the equation by $e^{\int P dx}$ and we have

$$e^{\int P dx}\frac{dy}{dx} + Pe^{\int P dx}y = Qe^{\int P dx} \quad . \quad . \quad . \quad (2)$$

$e^{\int P dx}$ is called the *integrating factor*, because it makes the left-hand side of (2) a complete differential, that is, it is the result of a direct differentiation, and therefore the integral may be inferred by inspection. The equation may now be written in the form

$$\frac{d}{dx}(e^{\int P dx}y) = Qe^{\int P dx}$$

and therefore $\quad e^{\int P dx}y = \int Qe^{\int P dx}dx + k$

that is $\quad y = e^{-\int P dx}[\int Qe^{\int P dx}dx + k] \,.\, (3)$

Solution of $\dfrac{dy}{dx} + Py = Q$ by another method.

First assume $Q = 0$ and we have

$$\frac{dy}{dx} + Py = 0$$

$$\therefore \frac{dy}{y} + Pdx = 0$$

$$\therefore \log y = -\int Pdx + k$$

$$\therefore y = Ke^{-\int P dx} \text{ where } K = e^k$$

Again suppose $K = f(x)$

$$\therefore y = f(x)e^{-\int P dx} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

On differentiating (4) with respect to $x$ we have

$$\frac{dy}{dx} = -Pf(x)e^{-\int P dx} + f'(x)e^{-\int P dx}$$

$$\therefore \frac{dy}{dx} + Pf(x)e^{-\int P dx} = f'(x)e^{-\int P dx}$$

that is $\quad \dfrac{dy}{dx} + Py = f'(x)e^{-\int P dx}$

On comparing this with the original equation, we see that

$$Q = f'(x)e^{-\int P dx}$$

$$\therefore f'(x) = Qe^{\int P dx}$$

and
$$f(x) = \int Qe^{\int P dx}dx + k$$

On substituting in (4) we have

$$y = e^{-\int P dx}[\int Qe^{\int P dx}dx + k]$$

*Example.*—Solution of

$$\frac{dy}{dx} + \frac{1}{x}y = x^2$$

Here $P = \dfrac{1}{x}$

$$\therefore \int P dx = \int \frac{dx}{x} = \log x$$

$$\therefore e^{\int P dx} = e^{\log x} = x \text{ and } Q = x^2$$

Substituting for P and Q in (3) we have

$$y = \frac{1}{x}\left[\int x^3 dx + k\right] \text{ since } e^{-\int P dx} = \frac{1}{x}$$

$$\therefore y = \frac{1}{x}\left[\frac{x^4}{4} + k\right] = \frac{x^3}{4} + \frac{k}{x}$$

*Example.*

$$\frac{dy}{dx} + \frac{x}{1+x^2}y = \frac{1}{2x(1+x^2)}$$

Here $P = \dfrac{x}{1+x^2}$, $\displaystyle\int P dx = \int \frac{x dx}{1+x^2} = \frac{1}{2}\log(1+x^2)$

and $e^{-\int P dx} = e^{-\frac{1}{2}\log(1+x^2)} = e^{\log(1+x^2)^{-\frac{1}{2}}} = \dfrac{1}{\sqrt{1+x^2}}$

and $e^{\int P dx} = \sqrt{1+x^2}$

$$Q = \frac{1}{2x(1+x^2)}$$

$$\therefore \int e^{\int P dx} Q dx = \int \frac{(1+x^2)^{\frac{1}{2}}dx}{2x(1+x^2)} = \frac{1}{2}\int \frac{dx}{x(1+x^2)^{\frac{1}{2}}}$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{x}\right)$$

$$\therefore y = (1+x^2)^{-\frac{1}{2}}\left[\frac{1}{2}\log\left(\frac{\sqrt{1+x^2}-1}{x}\right) + k\right]$$

*Example.*—Suppose the E.M.F. in an electric circuit to obey the law

$$E = E_0 \sin pt$$

and we require the law of the current, where there is self-induction in the circuit, then we have

$$E = RI + L\frac{dI}{dt}$$

where R is the resistance, I the current, L the coefficient of self-induction,

$$\therefore L\frac{dI}{dt} + RI = E_0 \sin pt$$

we have
$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E_0}{L} \sin pt$$

where $p$ is $2\pi \times$ frequency.

On comparing this with (1) (p. 258)

we have $\frac{R}{L}$ instead of P and $\frac{E_0}{L} \sin pt$ instead of Q

$$\therefore I = e^{-\int\frac{R}{L}dt}\left[\int e^{\int\frac{R}{L}dt}\frac{E_0}{L} \sin pt dt + k\right]$$

$$= e^{-\frac{Rt}{L}}\frac{E_0}{L}\left[\int e^{\frac{Rt}{L}} \sin pt dt + \frac{L}{E_0}k\right]$$

$$\therefore I = e^{-\frac{Rt}{L}}\frac{E_0}{L}\left[L^2 e^{\frac{Rt}{L}}\left\{\frac{\frac{R}{L} \sin pt - p \cos pt}{R^2 + L^2p^2}\right\} + \frac{L}{E_0}k\right]$$

$$= e^{-\frac{Rt}{L}} E_0\left[e^{\frac{Rt}{L}}\left\{\frac{R \sin pt - Lp \cos pt}{R^2 + L^2p^2}\right\} + \frac{k}{E_0}\right]$$

$$= E_0\left\{\frac{R \sin pt - Lp \cos pt}{R^2 + (Lp)^2}\right\} + ke^{-\frac{Rt}{L}} \quad . \quad . \quad (A)$$

Now $R = \sqrt{R^2 + (Lp)^2} \cos \theta$ (see Fig. 101)

and $Lp = \sqrt{R^2 + (Lp)^2} \sin \theta$ ,

and on substituting in (A) we have

$$I = E_0\sqrt{R^2 + (Lp)^2}\,\frac{(\sin pt \cos\theta - \cos pt \sin\theta)}{R^2 + (Lp)^2} + ke^{-\frac{Rt}{L}}$$

$$\therefore I = \frac{E_0}{\sqrt{R^2 + (Lp)^2}} \sin(pt - \theta) + ke^{-\frac{Rt}{L}}$$

This shows that the current lags behind the E.M.F. by an angle $\theta = \tan^{-1}\frac{Lp}{R}$

We see that by increasing the frequency or the inductance we increase the angle of lag, and that by diminishing the resistance the lag is increased. The term $ke^{-\frac{Rt}{L}}$ is called the *evanescent term* since it gets smaller and smaller as time goes on, but when $t$ is very small it may be an important term.

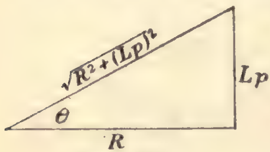

Fig. 101.

A linear equation of the form

$$\frac{dy}{dx} + Py = Qy^n \quad . \quad . \quad . \quad . \quad . \quad (1)$$

can be reduced to the form

$$\frac{dy}{dx} + Py = Q$$

by assuming $z = y^{1-n}$

From (1) we get

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)Py^{1-n} = (1-n)Q \quad (2)$$

$$\frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$$

Substituting in (2) we have

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q$$

which is a linear equation in $z$.

*Example.*—

$$\frac{dy}{dx}+\frac{1}{x}y = xy^4$$

Divide by $y^4$ and we have

$$y^{-4}\frac{dy}{dx}+\frac{1}{x}y^{-3} = x \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Let $z = y^{-3}$ $\quad\quad \therefore \frac{dz}{dx} = -3y^{-4}\frac{dy}{dx}$

Substituting in (3) we have

$$-\frac{1}{3}\frac{dz}{dx}+\frac{1}{x}z = x$$

$$\therefore \frac{dz}{dx}-\frac{3}{x}z = -3x$$

Here $P = -\frac{3}{x}$ and $Q = -3x$

$$\begin{aligned}
\int P dx &= -3\log x = \log x^{-3} \\
\therefore e^{\int P dx} &= x^{-3} \text{ and } e^{-\int P dx} = x^3 \\
\therefore z &= x^3[-\int x^{-3}\times 3x dx + k] \\
&= x^3[+3x^{-1}+k] \\
\therefore z &= 3x^2+kx^3 \\
\therefore y^{-3} &= 3x^2+kx^3
\end{aligned}$$

## Solution of Homogeneous Equations.

The equation $Mdx+Ndy = 0$ is said to be homogeneous when M and N are of the same degree and homogeneous functions of $x$ and $y$ In order to solve a homogeneous equation, assume $y = vx$ where $v$ and $x$ are both variable

*Example.*—

$$xdx+ydy = 2ydx \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $y = vx$ $\quad\quad \therefore dy = vdx+xdv$

On substituting in (1) for $dy$ and $y$ we have

$$xdx + vx(vdx + xdv) = 2vxdx$$
$$\therefore (1 - 2v + v^2)dx + vxdv = 0$$
$$\therefore \frac{dx}{x} + \frac{vdv}{(1-v)^2} = 0$$
$$\therefore \log x + \int \frac{vdv}{(1-v)^2} = k \quad . \quad . \quad (2)$$

Let $z = 1 - v \qquad \therefore dz = -dv$

$$\therefore \int \frac{vdv}{(1-v)^2} = -\int \frac{(1-z)dz}{z^2} = -\int \frac{dz}{z^2} + \int \frac{dz}{z}$$
$$= \frac{1}{z} + \log z$$
$$= \frac{1}{1-v} + \log(1-v).$$

therefore (2) becomes

$$\log x + \frac{1}{1-v} + \log(1-v) = k$$
$$\therefore \log x + \frac{x}{x-y} + \log\left(\frac{x-y}{x}\right) = k$$

**To solve** $\{x + \sqrt{y^2 + x^2}\}dy - ydx = 0$

Here let $y = vx$ therefore $dy = vdx + xdv$ therefore

$$\{x + x\sqrt{v^2+1}\}(vdx + xdv) - vxdx = 0$$

On dividing by $x$

$$(1 + \sqrt{v^2+1})(vdx + xdv) - vdx = 0$$
$$\therefore v\sqrt{v^2+1}dx + x(1 + \sqrt{v^2+1})dv = 0$$

On dividing by $vx\sqrt{v^2+1}$ we have

$$\frac{dx}{x} + \frac{dv}{v} + \frac{dv}{v\sqrt{v^2+1}} = 0$$

Integrating we have

$$\log x + \log v + \log \frac{v}{1+\sqrt{v^2+1}} = k$$

that is,
$$\log \frac{v^2x}{1+\sqrt{v^2+1}} = k$$
$$\therefore \frac{y^2}{x+\sqrt{x^2+y^2}} = e^k = \text{K say.}$$
$$\therefore y^2 = 2\text{K}x + \text{K}^2$$
K being an arbitrary constant.

**Equations of the form**
$$(ax+by+c)dx+(a'x+b'y+c')dy = 0$$
may generally be solved by assuming $x = x'-l$ and $y = y'-m$ and the transformed equation becomes
$$(ax'+by'-al-bm+c)dx'+(a'x'+b'y'-a'l-b'm+c')dy' = 0$$
Now let
$$al+bm = c$$
$$a'l+b'm = c'$$
and these equations will determine $l$ and $m$ and the transformed equation will become
$$(ax'+by')dx'+(a'x'+b'y')dy' = 0$$

This equation is integrable by the preceding example by assuming $y' = vx'$ since it is rendered homogeneous.

**To solve the equation**
$$(3x+y+7)dx+(2x+5y+9)dy = 0$$
Assuming $x = x'-l$ and $y = y'-m$ the equation becomes
$$(3x'+y'-3l-m+7)dx'+(2x'+5y'-2l-5m+9)dy' = 0$$
Let
$$3l+m = 7$$
$$2l+5m = 9$$
therefore $l = 2$ and $m = 1$
and the equation becomes
$$(3x'+y')dx'+(2x'+5y')dy' = 0$$
which is homogeneous.

Assume $y' = vx'$ and this equation becomes

$$(3 + 3v + 5v^2)dx' + x'(2 + 5v)dv = 0$$

$$\therefore \frac{dx'}{x'} + \frac{(2 + 5v)dv}{3 + 3v + 5v^2} = 0$$

$$\therefore \log x' + \int \frac{(2 + 5v)dv}{3 + 3v + 5v^2} = k$$

$$\therefore \log x' + \tfrac{1}{2} \log (3 + 3v + 5v^2) + \frac{1}{\sqrt{51}} \tan^{-1} \frac{(10v + 3}{\sqrt{51}} = k$$

On substituting in terms of $x$ and $y$ we have

$$\log (x + 2) + \frac{1}{2} \log \left\{3 + 3\left(\frac{y+1}{x+2}\right) + 5\left(\frac{y+1}{x+2}\right)^2\right\} + \frac{1}{\sqrt{51}} \tan^{-1} \frac{(10y + 3x + 16)}{(x + 2)\sqrt{51}} = k$$

Integrate the following differential equations.

1. $(a + y)xdy + (b - x)ydx = 0$
   *Ans.* $\log y^a x^b + y - x = k$
2. $xy(a^2 + x^2)dy - (b^2 + y^2)dx = 0$
   *Ans.* $(b^2 + y^2)^{a2}(a^2 + x^2) = Cx^2$
3. $\text{Sin}\, \theta \cos \phi d\theta - \cos \theta \sin \phi d\phi = 0$
   *Ans.* $\text{Cos}\, \phi = k \cos \theta$
4. $\text{Cot}\, x \operatorname{cosec}^2 ydy + \cot y \operatorname{cosec}^2 xdx = 0$
   *Ans.* $\text{Cot}\, x \cot y = k$
5. $(x - y)dx + xdy = 0$
   *Ans.* $x = ke^{-\frac{y}{x}}$
6. $\left(y - x \cos \frac{x}{y}\right)dy + y \cos \frac{x}{y} dx = 0$
   *Ans.* $y = ke^{-\sin \frac{x}{y}}$
7. $(3x - 7y + 7)dy + (7x - 3y + 3)dx = 0$
   *Ans.* $(x - y + 1)^2(x + y - 1)^5 = k$

8. $t(1-t^2)\dfrac{ds}{dt}+(2t^2-1)s = nt^3$

$$Ans. \quad s = nt + kt\sqrt{(1-t^2)}$$

9. $\dfrac{dy}{d\theta}+y\cos\theta = \sin\theta\cos\theta$

$$Ans. \quad y = \sin\theta + ke^{-\sin\theta}-1$$

10. $\dfrac{dy}{d\theta}+\dfrac{y}{1+\theta^2} = \dfrac{\tan^{-1}\theta}{1+\theta^2}$

$$Ans. \quad y = \tan^{-1}\theta + ke^{-\tan^{-1}\theta}-1$$

11. $\dfrac{dy}{dx}-\dfrac{my}{x} = 1+\dfrac{1}{x}$

$$Ans. \quad y = \frac{x}{1-m}+kx^m-\frac{1}{m}$$

12. $\dfrac{dy}{dx}+y\sec^2 x = \sec^2 x\tan x$

$$Ans. \quad y = \tan x + ke^{-\tan x}-1$$

13. $\dfrac{dy}{dx}+y-e^{-x} = 0$

$$Ans. \quad y = xe^{-x}+ke^{-x}$$

14. $\dfrac{dy}{dx}+\dfrac{2xy}{1+x^2} = \dfrac{4x^2}{1+x^2}$

$$Ans. \quad y = \frac{1}{1+x^2}\left\{\frac{4x^3}{3}+k\right\}$$

15. $\dfrac{dy}{dx}-\dfrac{ay}{1+x} = e^x(1+x)^a$

$$Ans. \quad y = (1+x)^a\{e^x+k\}$$

16. $\dfrac{dy}{dx}-\dfrac{y}{x} = \sqrt{x^2+y^2}$

$$Ans. \quad y+\sqrt{x^2+y^2} = kx e^x$$

17. $xdx+ydy = a(xdy-ydx)$

$$Ans. \quad a\tan^{-1}\frac{y}{x} = \log\sqrt{x^2+y^2}+k$$

18. $(x+y)dx = (y-x)dy$

*Ans.* $x^2+2xy-y^2 = k$

19. $(1+x^2)\dfrac{dy}{dx}+4xy = \dfrac{1}{(1+x^2)^2}$

*Ans.* $\tan^{-1} x = y(x^2+1)^2+k$

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CHAPTER XXIV

# SOLUTION OF LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS WHEN THE SECOND MEMBER IS O

THERE are $n$ different values of $y$ which satisfy the equation

$$\frac{d^ny}{dx^n} + a\frac{d^{n-1}y}{dx^{n-1}} + b\frac{d^{n-2}y}{dx^{n-2}} \dots py = 0 \quad . \quad (a)$$

in which $a \ b \dots . p$ are constants; and if $y_1 \ y_2 \dots . y_n$ represent the $n$ different values of $y$ then the complete value of $y$ is

$$y = k_1y_1 + k_2y_2 \dots . k_ny_n$$

that is, the complete value of $y$ which satisfies the equation is the sum of the $n$ distinct values of $y$ each multiplied by an arbitrary constant.

Rule for solving a differential equation with constant coefficients when the right-hand member is 0

Form the **auxiliary equation** by assuming

$$y = ke^{mx}$$

On differentiating $n$ times and substituting in $(a)$ we have

$$ke^{mx}\{m^n + am^{n-1} + bm^{n-2} \dots . p\} = 0$$

On rejecting the common factor $ke^{mx}$ we have

$$m^n + am^{n-1} + bm^{n-2} \dots . p = 0$$

the *auxilary equation.*

Let $r_1 \ r_2 \ r_3 \dots . r_n$ be the $n$ roots of this equation,

all real and unequal, suppose; then the $n$ *particular solutions* are

$$y = k_1 e^{r_1 x}, \quad y = k_2 e^{r_2 x} \ldots, \quad y = k_n e^{r_n x};$$

and the general solution is

$$y = k_1 e^{r_1 x} + k_2 e^{r_2 x} + k_3 e^{r_3 x} \ldots k_n e^{r_n x}$$

Again, suppose the auxiliary equation to have a pair of imaginary roots of the form $p \pm q\sqrt{-1}$ then the complete value of $y$ will contain a pair of terms of the form

$$k_1 e^{(p+q\sqrt{-1})x} + k_2 e^{(p-q\sqrt{-1})x}$$

which may be written

$$e^{px}\{k_1 e^{(q\sqrt{-1})x} + k_2 e^{-(q\sqrt{-1})x}\} \quad \ldots \quad (b)$$

and on expanding

$$e^{(q\sqrt{-1})x} \quad \text{and} \quad e^{-(q\sqrt{-1})x}$$

and collecting the terms we get, on substituting in $(b)$,

$$k_1 e^{px}(\cos qx + \sqrt{-1}\sin qx) + k_2 e^{px}(\cos qx - \sqrt{-1}\sin qx)$$
$$= e^{px}\{(k_1 + k_2)\cos qx + \sqrt{-1}(k_1 - k_2)\sin qx\}$$
$$= e^{px}\{\mathrm{M}\cos qx + \mathrm{N}\sin qx\}$$

where $\mathrm{M} = k_1 + k_2$ and $\mathrm{N} = \sqrt{-1}(k_1 - k_2)$ (*See p.* 72)

Again, suppose there exist a pair of equal roots of the auxiliary equation $r_1 = r_2$ for example; then the general solution would contain a pair of terms $k_1 e^{r_1 x} + k_2 e^{r_1 x}$ which may be written $(k_1 + k_2)e^{r_1 x}$ but this would reduce the $n$ constants to $n - 1$; therefore in order to get the general solution we assume the two roots to differ by a finite small quantity $c$ and find the result as $c$ approaches 0

Thus $$k_1 e^{r_1 x} + k_2 e^{(r_1 + c)x} = e^{r_1 x}(k_1 + k_2 e^{cx})$$
$$= e^{r_1 x}\left(k_1 + k_2 + k_2 cx + k_2 \frac{(cx)^2}{\underline{|2}} + \text{etc.}\right)$$

and if we neglect the terms involving $c^2$ and higher powers it becomes

$$e^{r_1 x}(M + Nx)$$

where $M = k_1 + k_2$ and $N = k_2 c$

In like manner it may be shown that if the $n$ roots are each equal to $r_1$ the solution is

$$y = e^{r_1 x}(M + Nx + Qx^2 \ldots Vx^{n-1}).$$

**To solve** $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 16y = 0$

Here the auxiliary equation is

$$m^2 - 10m + 16 = 0$$

the roots of which are 8 and 2; therefore the complete solution is

$$y = k_1 e^{8x} + k_2 e^{2x}$$

the two particular solutions being

$$y = k_1 e^{8x} \quad \text{and} \quad y = k_2 e^{2x}$$

**To solve** $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 0$

Here the auxiliary equation is

$$m^2 - 6m + 13 = 0$$

the roots of which are $3 + 2\sqrt{-1}$ and $3 - 2\sqrt{-1}$ therefore the complete solution is

$$y = e^{3x}(k_1 \cos 2x + k_2 \sin 2x)$$

Observe that in this example the real part of the root is the coefficient of $x$ in the index of $e$ and the coefficient of $\sqrt{-1}$ is the multiple of the angle $x$

**To solve the equation**

$$\frac{d^5y}{dx^5} - 4\frac{d^4y}{dx^4} + 10\frac{d^3y}{dx^3} - 12\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0$$

Here the auxiliary equation is

$$m^5 - 4m^4 + 10m^3 - 12m^2 + 5m = 0$$

the roots of which are $0$, $1$, $1$, $1-2\sqrt{-1}$, and $1+2\sqrt{-1}$; therefore the complete solution is

$$y = k_1 + e^x(k_2 + k_3x) + e^x(k_4 \cos 2x + k_5 \sin 2x)$$

**To solve** $\dfrac{d^2y}{dx^2} + a^2y = 0$

Here the auxiliary equation is

$$m^2 + a^2 = 0$$
$$\therefore m = \pm a\sqrt{-1}$$

therefore
$$\begin{aligned} y &= k_1 \cos ax + k_2 \sin ax \\ &= \text{K} \sin (ax + \theta) \end{aligned}$$

where $\text{K} = \sqrt{k_1^2 + k_2^2}$ and $\tan\theta = \dfrac{k_1}{k_2}$

The equations $\dfrac{d^2y}{dx^2} + a^2y = \sin bx$ and $\dfrac{d^2y}{dx^2} + a^2y = \cos bx$ whose right-hand members are *not* 0 may be reduced to those whose right-hand members *are* 0 by differentiating and eliminating the sine or cosine function.

Given $\dfrac{d^2y}{dx^2} + a^2y = \sin bx$ on differentiating twice we have

$$\frac{d^4y}{dx^4} + a^2\frac{d^2y}{dx^2} = -b^2 \sin bx$$

and from the given equation, by substituting for $\sin bx$ we get

$$\frac{d^4y}{dx^4} + (a^2 + b^2)\frac{d^2y}{dx^2} + a^2b^2y = 0$$

Forming the auxiliary equation, we have

$$m^4 + (a^2 + b^2)m^2 + a^2b^2 = 0$$

therefore $m = \pm a\sqrt{-1}$ or $m = \pm b\sqrt{-1}$

Hence the complete solution is

$$y = k_1e^{(a\sqrt{-1})x} + k_2e^{(-a\sqrt{-1})x} + k_3e^{(b\sqrt{-1})x} + k_4e^{(-b\sqrt{-1})x}$$

which reduces to

$$y = (k_1 + k_2) \cos ax + (k_1 - k_2)\sqrt{-1} \sin ax$$
$$+ (k_3 + k_4) \cos bx + (k_3 - k_4)\sqrt{-1} \sin bx$$

or $y = M \cos ax + N \sin ax + M_1 \cos bx + N_1 \sin bx$

where $M = k_1 + k_2$, $N = (k_1 - k_2)\sqrt{-1}$, $M_1 = k_3 + k_4$ and $N_1 = (k_3 - k_4)\sqrt{-1}$

*Examples.*—

1. $\dfrac{dy}{dx} - ay = 0$

*Ans.* $y = ke^{ax}$

2. $\dfrac{dy}{dx} + by = 0$

*Ans.* $y = ke^{-bx}$

3. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 0$

*Ans.* $y = ke^{-4x} + k_1e^{-x}$

4. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$

*Ans.* $y = e^{-3x}(k + k_1x)$

5. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 13y = 0$

*Ans.* $y = e^{-3x}\{k \cos 2x + k_1 \sin 2x\}$

6. $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = 0$

*Ans.* $y = e^{-x}(k + k_1x + k_2x^2)$

7. $\dfrac{d^4y}{dx^4} + 4\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + y = 0$

*Ans.* $y = e^{-x}(k + k_1x + k_2x^2 + k_3x^3)$

8. $\dfrac{d^2y}{dx^2} + 3y = \sin \dfrac{2\pi x}{T}$

*Ans.* $y = k \cos \sqrt{3}x + k_1 \sin \sqrt{3}x + k_2 \cos \dfrac{2\pi x}{T}$

$$+ k_3 \sin \frac{2\pi x}{T}$$

9. $\dfrac{d^2y}{dx^2} + 4y = \cos\dfrac{2\pi x}{T}$

*Ans.* $y = k\cos 2x + k_1 \sin 2x + k_2 \cos\dfrac{2\pi x}{T}$
$+ k_3 \sin\dfrac{2\pi x}{T}$

10. $\dfrac{d^2x}{dt^2} + 2f\dfrac{dx}{dt} + n^2x = 0$

when $f = 0$ . . . . . . (a)
,, $f = n$ . . . . . . (b)
,, $f < n$ . . . . . . (c)
and ,, $f > n$ . . . . . . (d)

*Ans.* $x = k\sin nt + k_1\cos nt$ . . (a)
$x = Ae^{-nt}(k + k_1t)$ . . . . (b)
$x = Ae^{-ft}(k\cos gt + k_1\sin gt)$ (c)
where $g = \sqrt{n^2 - f^2}$
$x = Ae^{-ft}(ke^{gt} + k_1e^{-gt})$. . . (d)
where $g = \sqrt{f^2 - n^2}$

11. $\dfrac{d^5y}{dx^5} + 2\dfrac{d^2y}{dx^2} - 9\dfrac{dy}{dx} + 6y = 0$

*Ans.* $y = e^x(k + k_1x) + k_2e^{-2x}$
$+ A\sin\sqrt{3}x + B\cos\sqrt{3}x$

12. $\dfrac{d^2x}{dt^2} + n^2x = a\sin pt$

*Ans.* $x = k_1\cos nt + k_2\sin nt + k_3\cos pt + k_4\sin pt$

13. $\dfrac{d^2x}{dt^2} + 2f\dfrac{dx}{dt} + n^2x = a\sin pt$

*Ans.* $x = Ae^{-ft}\{k_1\cos\sqrt{n^2 - f^2}t + k_2\sin\sqrt{n^2 - f^2}t\}$
$+ k_3\cos pt + k_4\sin pt$
$n^2 - f^2$ being positive.

CHAPTER XXV

# SOLUTION OF DIFFERENTIAL EQUATIONS BY SYMBOLICAL METHODS

If D be substituted for the differential operator $\frac{d}{dx}$ in the differential equation

$$\frac{d^n y}{dx^n} + A_1\frac{d^{n-1}y}{dx^{n-1}} + A_2\frac{d^{n-2}y}{dx^{n-2}} \ldots A_n y = 0 \,. \quad (1)$$

we have

$$(D^n + A_1 D^{n-1} + A_2 D^{n-2} \ldots A_n)y = 0$$

which may be written thus

$$F(D)y = 0$$

$F(D) = 0$ may be regarded as an ordinary algebraic equation. Let the roots of $F(D)$ be $m_1 \quad m_2 \quad m_3 \ldots m_n$ therefore (1) may be written thus

$$(D - m_1)(D - m_2)(D - m_3) \ldots (D - m_n)y = 0 \quad (2)$$

The complete solution of (2) is obviously made up of the sum of the solutions of

$$\begin{aligned} (D - m_1)y &= 0 \\ (D - m_2)y &= 0 \\ &: : \\ \text{and } (D - m_n)y &= 0 \end{aligned}$$

The solution of $(D - m_1)y = 0$

that is $$y = \frac{0}{D - m_1} = (D - m_1)^{-1}0$$

is obtained in the following way

$$(D - m_1)y = 0$$

may be written thus

$$\left(\frac{d}{dx} - m_1\right)y = 0$$

that is
$$\frac{dy}{dx} - m_1 y = 0$$

$$\therefore \frac{dy}{y} = m_1 dx$$

$$\log y = m_1 x + c$$

$$\therefore y = e^{m_1 x + c} = e^c \times e^{m_1 x}$$

$$\therefore y = k_1 e^{m_1 x}$$

The complete solution of (1) is therefore

$$y = k_1 e^{m_1 x} + k_2 e^{m_2 x} \ldots k_n e^{m_n x}$$

The symbol D may be regarded as an ordinary algebraic quantity and it obeys the laws of algebra.

(1) The *distributive law*

(2) The *index law*

(3) The *commutative law* in regard to constants only.

$$D(u + v + w \ldots) = Du + Dv + Dw \ldots$$

the distributive law.

The index law is

$$D^m \times D^n f(x) = D^{m+n} f(x)$$

where $n$ and $m$ are integers.

The commutative law is

$$D^n(auv) = aD^n(uv)$$

where $a$ is a constant.

$D^n e^{ax}$ denotes that $e^{ax}$ is to be differentiated $n$ times

$$\therefore D^n e^{ax} = a^n e^{ax}$$

Similarly
$$f(D)e^{ax} = f(a)e^{ax}$$

$$(D^3 + 4D^2 + 3D + 1)e^{ax} = (a^3 + 4a^2 + 3a + 1)e^{ax}$$

Let $D^{-n}$ denote such an operation that

$$D^{-n}D^{n}e^{ax} = e^{ax}$$

Therefore $D^{-n}$ means the inverse of $D^n$ that is $D^{-n}$ means integrate $n$ times

$$D^{-n}e^{ax} = \frac{e^{ax}}{D^n} = \frac{e^{ax}}{a^n}$$

We have $\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$

This method fails when $a$ is a root of

$$f(D) = 0$$

For suppose $a$ to be a root of $f(D) = 0$ then

$$\frac{1}{f(D)} = \frac{e^{ax}}{f(a)} = \frac{e^{ax}}{0} = \infty$$

since $f(a) = 0$

If $a$ be a root of $f(D)$ then $D-a$ must be a factor of $f(D)$

$$\therefore f(D) = (D-a)F(D)$$

$$\therefore \frac{1}{f(D)}e^{ax} = \frac{1}{D-a} \times \frac{1}{F(D)}e^{ax} = \frac{1}{D-a} \times \frac{e^{ax}}{F(a)}$$

$$= \frac{xe^{ax}}{F(a)}$$

Since $(D-a)^{-1}e^{ax} = xe^{ax}$

No constant being added since we are only seeking a particular integral.

Let X be any function of $x$ then

$$F(D)e^{ax}X = e^{ax}F(D+a)X$$

That is, we may bring $e^{ax}$ to the left of $F(D)$ provided we substitute $D+a$ for D

$$De^{ax}X = ae^{ax}X + e^{ax}DX = e^{ax}(D+a)X$$

Similarly it may be shown that

$$D^{n}e^{ax}X = e^{ax}(D+a)^{n}X \quad . \quad . \quad . \quad (1)$$

Let $X = (D+a)^n Z$ where Z is any function of $x$ then $Z = (D+a)^{-n}X$ and by (1) we have

$$D^n e^{ax}Z = e^{ax}(D+a)^n Z$$
$$\therefore D^n e^{ax}(D+a)^{-n}X = e^{ax}X$$
$$\therefore D^{-n}e^{ax}X = e^{ax}(D+a)^{-n}X$$

This shows that (1) holds good when $n$ is a negative integer.

*Example.*—Solve $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+4y = e^{2x}$

Here we have $(D^2+3D+4)y = e^{2x}$

$$\therefore y = \frac{e^{2x}}{D^2+3D+4} = \frac{e^{2x}}{2^2+3\times 2+4} = \frac{e^{2x}}{14}$$

the particular integral. See the following chapter.

*Example.*—Solve $\dfrac{D+2}{(D+3)(D+7)}e^{4x}$

where D stands for $\dfrac{d}{dx}$

Since $f(D)e^{ax} = f(a)e^{ax}$

we have $\dfrac{D+2}{(D+3)(D+7)}e^{4x} = \dfrac{6}{7\times 11}e^{4x}$

*Example.*—Solve $\dfrac{e^x x}{D^3-3D^2+3D-1} = (D-1)^{-3}e^x x$

Since $F(D)e^{ax}X = e^{ax}F(D+a)X$

we have $(D-1)^{-3}e^x x = e^x(D+1-1)^{-3}x$

$$= e^x D^{-3}x = \frac{e^x x^4}{2\times 3\times 4} = \frac{e^x x^4}{24}$$

*Example.*—Solve

$$\frac{1}{(D-3)^3}e^{3x}\cos x = e^{3x}\frac{\cos x}{D^3} = -e^{3x}\sin x$$

## CHAPTER XXVI

# SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS WHEN THE RIGHT-HAND MEMBER IS A FUNCTION OF $x$

THE general form of these equations is

$$\frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} \ldots A_n y = X \quad (1)$$

where $A_1 \; A_2 \; A_3 \ldots A_n$ and $X$ are constants or functions of $x$

The complete solution of (1) consists of two parts called the **particular integral** and the **complementary function**.

Let $f(x)$ be any particular value of $y$ which satisfies the equation; and let

$$y = V + f(x)$$

Substituting this value of $y$ in (1) we have

$$\frac{d^n V}{dx^n} + A_1 \frac{d^{n-1} V}{dx^{n-1}} + A_2 \frac{d^{n-2} V}{dx^{n-2}} \ldots A_n V = 0 \quad (2)$$

Suppose

$$V = V_1, \; V = V_2, \; V = V_3 \ldots V = V_n$$

to be solutions of (2) then

$$V = k_1 V_1 + k_2 V_2 + k_3 V_3 \ldots k_n V_n \quad . \quad (3)$$

is the general solution of (2) containing $n$ arbitrary constants, and therefore

$$y = k_1 V_1 + k_2 V_2 + k_3 V_3 \ldots k_n V_n + f(x)$$

is the most general solution of (1)

The right-hand member of (3) is called the **complementary function** and is the primitive of (1) when the right-hand member is made zero. It contains $n$ arbitrary constants. The **particular integral** of (1) is $f(x)$

*Example.*

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = e^x \quad . \quad . \quad (a)$$

Here the auxiliary equation of $(a)$ when the right-hand member is assumed to be zero is

$$m^3 + 2m^2 - 5m - 6 = 0$$
$$\therefore m = -1, \ 2, \ -3$$

The complementary function is therefore

$$k_1e^{-x} + k_2e^{2x} + k_3e^{-3x}$$

and the particular integral is $-\dfrac{e^x}{8}$

The complete solution is

$$y = k_1e^{-x} + k_2e^{2x} + k_3e^{-3x} - \frac{e^x}{8}$$

Solution of $\dfrac{d^ny}{dx^n} + A_1\dfrac{d^{n-1}y}{dx^{n-1}} + A_2\dfrac{d^{n-2}y}{dx^{n-2}} \ldots A_ny = f(x)$ by the symbolical method.

Here we have $\quad F(D)y = f(x)$

the particular integral may be expressed thus

$$y = \frac{1}{F(D)}f(x) \quad . \quad . \quad . \quad . \quad . \quad (a)$$

The meaning to be attached to $(a)$ is such that when it is operated upon by $F(D)$ the result is

$$f(x)$$

Therefore the operator $\dfrac{1}{F(D)}$ is the inverse of the operator $F(D)$

*Example.*—Solve $(D^3 + 3D^2 + 2D)y = x^3$

Here the roots of $f(D)$ are

$$D = 0, \; D = -2 \text{ and } D = -1$$

The complementary function is $k_1 + k_2e^{-x} + k_3e^{-2x}$ and the particular integral

$$= \frac{1}{D^3 + 3D^2 + 2D}x^3 = \frac{1}{2D}\left(1 + \frac{3D}{2} + \frac{D^2}{2}\right)^{-1}x^3$$

$$= \frac{1}{2D}\left(1 - \frac{3}{2}D + \frac{7}{4}D^2 - \frac{15}{8}D^3 \ldots\right)x^3$$

$$= \frac{1}{2D}\left(x^3 - \frac{9}{2}x^2 + \frac{21}{2}x - \frac{45}{4}\right)$$

$$= \frac{1}{2}\left(\frac{x^4}{4} - \frac{3x^3}{2} + \frac{21x^2}{4} - \frac{45x}{4}\right)$$

The complete solution is

$$y = k_1 + k_2e^{-x} + k_3e^{-2x} + \frac{x}{8}\left(x^3 - 6x^2 + 21x - 45\right)$$

$D \sin pt = p \cos pt$ where $D$ stands for $\dfrac{d}{dt}$

$$D^2 \sin pt = Dp \cos pt = -p^2 \sin pt$$
$$D^3 \sin pt = D \times D^2 \sin pt = -Dp^2 \sin pt$$
$$D^4 \sin pt = D^2 \times D^2 \sin pt = -D^2p^2 \sin pt = p^4 \sin pt$$

and

$$D^5 \sin pt = p^4 D \sin pt$$

$$\begin{aligned}(D + a) \sin pt &= D \sin pt + a \sin pt \\ &= p \cos pt + a \sin pt \\ &= \sqrt{a^2 + p^2} \sin (pt + \theta) \text{ where } \tan \theta = \frac{p}{a}\end{aligned}$$

This is called a direct operation on $\sin pt$

The inverse operation $(D + a)^{-1} \sin pt$ is performed as follows.

We have $$(D + a)^{-1} \sin pt = \frac{\sin pt}{a + D}$$

$$= \frac{a - D}{a^2 - D^2} \sin pt = (a - D)\frac{\sin pt}{a^2 + p^2}$$

$$= \frac{a \sin pt - p \cos pt}{a^2 + p^2}$$

$$= \frac{\sin (pt - \theta)}{\sqrt{a^2 + p^2}} \text{ where } \tan \theta = \frac{p}{a}$$

The complete value of $(D + a)^{-1} \sin pt$ is

$$\frac{\sin (pt - \theta)}{\sqrt{a^2 + p^2}} + ke^{-at}$$

It should be noted that in any operation performed upon $\sin pt$ we can substitute $-p^2$ for $D^2$

therefore $$f(D^2) \sin pt = f(-p^2) \sin pt$$
and $$f(D^2) \cos pt = f(-p^2) \cos pt$$
where $$D = \frac{d}{dt}$$

*Example.*— $$\frac{a + bD + cD^2 + eD^3}{l + mD + nD^2 + rD^3} \sin pt$$

$$= \frac{a - cp^2 + (b - ep^2)D}{l - np^2 + (m - rp^2)D} \sin pt$$

$$= \frac{A + BD}{L + MD} \sin pt$$

$$= \sqrt{\frac{A^2 + (Bp)^2}{L^2 + (Mp)^2}} \sin (pt + \theta - \phi) + ke^{-\frac{Lt}{M}}$$

where $$A = a - cp^2 \text{ and } B = b - ep^2$$
$$L = l - np^2 \text{ and } M = m - rp^2$$
$$\tan \theta = \frac{Bp}{A} \text{ and } \tan \phi = \frac{Lp}{M}$$

The direct operation $(a + b\mathrm{D}) \sin pt$ is performed by multiplying by

$$\sqrt{a^2 + (bp)^2}$$

and adding the angle $\theta = \tan^{-1}\dfrac{bp}{a}$ to $pt$

thus $$(a + b\mathrm{D}) \sin pt = \sqrt{a^2 + (bp)^2} \sin (pt + \theta)$$

The inverse operation

$$\frac{\sin pt}{\alpha + \beta\mathrm{D}}$$

is performed by dividing by

$$\sqrt{\alpha^2 + (\beta p)^2}$$

and subtracting the angle $\phi = \tan^{-1}\dfrac{\beta p}{\alpha}$

and adding the evanescent term

$$ke^{-\frac{\alpha t}{\beta}}$$

thus $$\frac{\sin pt}{\alpha + \beta\mathrm{D}} = \frac{\sin (pt - \phi)}{\sqrt{\alpha^2 + \beta^2 p^2}} + ke^{-\frac{\alpha t}{\beta}}$$

*Example.*—Integrate $\int e^{ax} \sin bx dx$

Here we have $\mathrm{D}^{-1}e^{ax} \sin bx = e^{ax}(\mathrm{D} + a)^{-1} \sin bx$

$$= e^{ax}\frac{\sin bx}{\mathrm{D} + a} = e^{ax}\frac{a - \mathrm{D}}{a^2 - \mathrm{D}^2} \sin bx$$

$$= e^{ax}\left\{\frac{a \sin bx - b \cos bx}{a^2 + b^2}\right\}$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin (bx - \theta) \quad \text{where } \theta = \tan^{-1}\frac{b}{a}$$

$$\mathrm{D}^{-1}e^{\frac{\mathrm{R}t}{\mathrm{L}}} \sin pt = e^{\frac{\mathrm{R}t}{\mathrm{L}}}\left(\mathrm{D} + \frac{\mathrm{R}}{\mathrm{L}}\right)^{-1} \sin pt$$

$$= e^{\frac{\mathrm{R}t}{\mathrm{L}}}\frac{\sin pt}{\frac{\mathrm{R}}{\mathrm{L}} + \mathrm{D}} = e^{\frac{\mathrm{R}t}{\mathrm{L}}}\frac{\left(\frac{\mathrm{R}}{\mathrm{L}} - \mathrm{D}\right)}{\frac{\mathrm{R}^2}{\mathrm{L}^2} - \mathrm{D}^2} \sin pt$$

$$Le^{\frac{Rt}{L}}\frac{(R - LD)\sin pt}{R^2 + (Lp)^2} + ke^{-\frac{Rt}{L}}$$

$$= L\frac{e^{\frac{Rt}{L}}\sin(pt - \theta)}{\sqrt{R^2 + (Lp)^2}} + ke^{-\frac{Rt}{L}}$$

Let there be a circuit with inductance L and resistance R and a condenser in series. Let the E.M.F. obey the law $E = E_0 \sin pt$

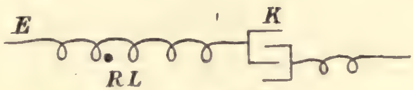


FIG. 102.

Let K denote the capacity of the condenser, then the rate of charging

$$I = \frac{dQ}{dt} = \frac{d}{dt}KE = KDE$$

$$\therefore I = \frac{E}{\dfrac{1}{KD}}$$

$$\text{Now } E = RI + LDI + \frac{I}{KD}$$

$$\therefore I = \frac{E}{R + LD + \dfrac{1}{KD}} \quad \text{where } D = \frac{d}{dt}$$

$$\therefore I = \frac{KDE_0 \sin pt}{1 + RKD + LKD^2}$$

$$= \frac{KE_0D \sin pt}{1 - LKp^2 + RKD}$$

$$= \frac{KE_0p \cos pt}{1 - LKp^2 + RKD} = \frac{KE_0p \sin\left(pt + \frac{\pi}{2}\right)}{1 - LKp^2 + RKD}$$

$$= \frac{KE_0 p \sin\left(pt + \frac{\pi}{2} - \theta\right)}{\sqrt{(1 - LKp^2)^2 + (RKp)^2}}$$

when $\tan\theta = \dfrac{RKp}{1 - LKp^2}$

*Example.*

$$\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = f(x)$$

Here we have

$$\{D^2 + (a+b)D + ab\}y = f(x)$$

$$(D+a)(D+b)y = f(x)$$

$$\therefore y = (D+a)^{-1}(D+b)^{-1}f(x) + (D+a)^{-1}(D+b)^{-1} \times 0$$

$$= \frac{1}{a-b}\{(D+b)^{-1} - (D+a)^{-1}\}f(x) + k_1e^{-ax} + k_2e^{-bx}$$

$$= \frac{1}{a-b}\left[e^{-bx}\int e^{bx}f(x)dx - e^{-ax}\int e^{ax}f(x)dx\right]$$

$$+ k_1e^{-ax} + k_2e^{-bx}$$

*Example.*

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = \sin x$$

Here we have $(D+3)(D+4)y = \sin x$

$$\therefore y = (D+3)^{-1}\sin x - (D+4)^{-1}\sin x + (D+3)^{-1} \times 0$$

$$-(D+4)^{-1} \times 0$$

$$= \frac{3\sin x - \cos x}{10} - \frac{4\sin x - \cos x}{17} + k_1e^{-3x} + k_2e^{-4x}$$

*Examples.*—

Solve by the symbolical method the following examples.

1. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$ *Ans.* $y = ke^x + k'e^{2x}$

2. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 13y = 0$

*Ans.* $y = ke^{2x}\cos 3x + k'e^{2x}\sin 3x$

3. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 0$ *Ans.* $y = e^x(k + k'x)$

4. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$

*Ans.* $y = e^x(k + k'x + k''x^2)$

5. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = 0$

*Ans.* $y = ke^{-x} + e^x(k' + k''x)$

6. $\dfrac{d^4y}{dx^4} + 2a^2\dfrac{d^2y}{dx^2} + a^4y = 0$

*Ans.* $y = (k + k'x)\cos ax + (k'' + k'''x)\sin ax$

7. $\dfrac{d^2y}{dx^2} + a^2y = \cos ax$

*Ans.* $y = k\cos ax + \left(k' + \dfrac{x}{2a}\right)\sin ax$

8. $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + a^2y = e^x$

*Ans.* $y = (k + k'x)e^{ax} + \dfrac{e^x}{(a-1)^2}$

9. $(D^2 - 1)y = 5x + 2$

*Ans.* $y = k_1e^x + k_2e^{-x} - (5x + 2)$

10. $(D^2 + 2D + 1)y = 2e^{2x}$

*Ans.* $y = e^{-x}(k + k_1x) + \frac{2}{9}e^{2x}$

12. $(D^3 - D^2 - 8D + 12)y = 0$

*Ans.* $e^{2x}(k + k_1x) + k_2e^{-3x}$

13. $(D^3 - 1)y = (e^x + 1)^2$

*Ans.* $y = e^{-.5x}\left(k\cos\dfrac{\sqrt{3}}{2}x + k_1\sin\dfrac{\sqrt{3}}{2}x\right)$

$+ e^x\left(k_2 + \dfrac{2}{3}x\right) + \dfrac{e^{2x}}{7} - 1$

14. $(D - 1)^2y = 3e^{2\cdot 5x}$

*Ans.* $y = e^x(k + k_1x) + \frac{4}{3}e^{2\cdot 5x}$

15. $(D^2 - 5D + 6)y = e^{ax} + x$

$$Ans.\quad y = ke^{2x} + k_1e^{3x} + \frac{x}{6} + \frac{5}{36} + \frac{e^{ax}}{a^2 - 5a + 6}$$

16. $(D^4 - 2D^3 + D^2)y = x$

$$Ans.\quad y = k + k_1x + e^x(k_2 + k_3x) + x^2 + \frac{x^3}{6}$$

## CHAPTER XXVII

# SOLUTION OF HOMOGENEOUS LINEAR EQUATIONS

The general form of a homogeneous linear equation is

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}\ldots a_ny = f(x)\,. \quad (1)$$

where $a_1, a_2, \ldots a_n$ are constants and $f(x)$ is a function of $x$

By assuming $x = e^z$ the equation can be transformed into one with constant coefficients.

Now $$z = \log x,\quad \frac{dz}{dx} = \frac{1}{x}$$

and $$\frac{dy}{dx} = \frac{dy}{dz}\times\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz}$$

Again $$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}-\frac{dy}{dz}$$

Similarly $$x^3\frac{d^3y}{dx^3} = \frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}$$

and $$x^n\frac{d^ny}{dx^n} = \frac{d^ny}{dz^n}-\frac{n(n-1)}{2}\frac{d^{n-1}y}{dz^{n-1}}\ldots$$
$$+(-1)^{n-1}\lfloor\underline{n-1}\,\frac{dy}{dz}$$

On substituting D for $x\frac{d}{dx}$ or $\frac{d}{dz}$ we have

$$
\begin{aligned}
x\frac{dy}{dx} &= \mathrm{D}y \\
x^2\frac{d^2y}{dx^2} &= \mathrm{D}(\mathrm{D}-1)y \\
x^3\frac{d^3y}{dx^3} &= \mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)y \\
&\cdots\cdots \\
x^n\frac{d^ny}{dx^n} &= \mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)\ldots(\mathrm{D}-n+1)y
\end{aligned}
$$

Substituting $e^z$ for $x$ in (1) transforms it into

$$
\begin{aligned}
\{\mathrm{D}(\mathrm{D}-1)\ldots(\mathrm{D}-n+1)+a_1\mathrm{D}(\mathrm{D}-1)\ldots(\mathrm{D}-n+2) \\
+\ldots a_n\}y &= \mathrm{F}(z) \\
\therefore f(\mathrm{D})y &= \mathrm{F}(z) \quad . \quad . \quad . \quad . \quad (2)
\end{aligned}
$$

This equation may be solved by the methods given in the preceding chapter.

Suppose $y = f(z)$ to be a solution of (2)
then $\quad y = f(\log x)$ since $z = \log x$

*Example.*

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y = \log x$$

Let $x = e^z$ and we have

$$
\begin{aligned}
\{\mathrm{D}(\mathrm{D}-1)+\mathrm{D}+1\}y &= z \\
\therefore (\mathrm{D}^2+1)y &= z \\
\therefore y &= (1+\mathrm{D}^2)^{-1}\times 0+(1+\mathrm{D}^2)^{-1}z \\
&= k_1\cos z+k_2\sin z+z \\
\therefore y &= k\cos(\log x)+k_2\sin(\log x)+\log x
\end{aligned}
$$

The complete solution of

$$x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots a_ny = f(x) \quad . \quad (1)$$

contains the *complementary function* and the *particular integral.*

The complementary function being the complete solution of (1) when we assume the right-hand member zero, and it contains $n$ arbitrary constants. The particular integral is that function of $x$ not containing an arbitrary constant which satisfies the equation.

*Example.*

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^3$$

Here we have

$$\{D(D-1) + 4D + 2\}y = e^{3z}$$

where $$D = \frac{d}{dz} \text{ and } x = e^z$$

$$\therefore (D^2 + 3D + 2)y = e^{3z}$$
$$\therefore (D + 2)(D + 1)y = e^{3z}$$

The complementary function is the solution of

$$(D + 2)(D + 1)y = 0$$

$$\therefore y = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) \times 0$$

that is $$y = ke^{-z} + k_1e^{-2z}$$

and the particular integral is

$$y = \frac{e^{3z}}{(D+2)(D+1)} = \frac{e^{3z}}{(3+2)(3+1)} = \frac{e^{3z}}{20}$$

therefore the complete solution is

$$y = \frac{k}{x} + \frac{k_1}{x^2} + \frac{x^3}{20}$$

*Example.*

$$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$$

On substituting $x = e^z$ and D for $\frac{d}{dz}$ we have

$$\{D(D-1)(D-2) + 3D(D-1) + D + 1\}y = e^z$$

that is $$(D^3+1)y = e^z$$

$$\therefore y = (1+D^3)^{-1}\times 0 + \frac{e^z}{D^3+1}$$

$$y = ke^{-z} + e^{-\frac{z}{2}}\left\{k_2 e^{\frac{\sqrt{3}}{2}z} + k_3 e^{-\frac{\sqrt{3}}{2}z}\right\} + \frac{e^z}{2}$$

$$= ke^{-z} + e^{-\frac{z}{2}}\left\{k_2 \cos\frac{\sqrt{3}}{2}z + k_3 \sin\frac{\sqrt{3}}{2}z\right\} + \frac{e^z}{2}$$

$$\therefore y = \frac{k}{x} + \frac{1}{\sqrt{x}}\left\{k_2 \cos\frac{\sqrt{3}}{2}\left(\log x\right) + k_3 \sin\frac{\sqrt{3}}{2}\left(\log x\right)\right\} + \frac{x}{2}$$

If D be substituted for $x\dfrac{d}{dx}$ in the equation

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots a_ny = f(x) \quad . \quad (1)$$

we have $$F(D)y = f(x)$$

Let the roots of $F(D) = 0$ be $m_1, m_2, \ldots m_n$ and we have

$$(D-m_1)(D-m_2)\ldots(D-m_n)y = f(x)$$

$$\therefore y = \frac{1}{(D-m_1)(D-m_2)\ldots(D-m_n)}f(x)$$

that is

$$y = \left(\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2}\ldots\frac{A_n}{D-m_n}\right)f(x)$$

where $A_1, A_2, \ldots$ are constants.

Now $\dfrac{A_n}{D-m_n}f(x)$ may be written thus

$$x\frac{dy}{dx} - m_ny = f(x)$$

$$\therefore \frac{dy}{dx} - \frac{m_n}{x}y = \frac{1}{x}f(x)$$

This is a linear equation the solution of which is represented by

$$y = x^{m_n}\int x^{-m_n-1}f(x)dx$$

and therefore the particular integral of (1) is given by

$$y = A_1x^{m_1}\int x^{-m_1-1}f(x)dx \; . \; . \; . \; A_nx^{m_n}\int x^{-m_n-1}f(x)dx$$

*Example.*

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x^2$$

Here we have

$$\{D(D-1) + 4D + 2\}y = x^2$$
$$\therefore (D^2+3D+2)y = x^2$$
$$\therefore y = \left(\frac{1}{D+1} - \frac{1}{D+2}\right)x^2$$
$$\therefore y = \frac{1}{D+1}x^2 - \frac{1}{D+2}x^2$$
$$= x^{-1}\int x^2dx - x^{-2}\int x^3dx = \frac{x^2}{3} - \frac{x^2}{4}$$
$$= \frac{x^2}{12}$$

the particular integral.

The complementary function is $y = \dfrac{k}{x} + \dfrac{k_1}{x^2}$ and the complete solution is

$$y = \frac{k}{x} + \frac{k_1}{x^2} + \frac{x^2}{12}$$

*Examples.*

1. $x^2\dfrac{d^2y}{dx^2} + 6x\dfrac{dy}{dx} + 4y = x^2$

*Ans.* $y = kx^{-1} + k_1x^{-4} + \dfrac{x^2}{18}$

2. $x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 5y = x^3$

*Ans.* $y = kx^{-1} + k_1x^{-5} + \frac{x^3}{32}$

3. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x^2$

*Ans.* $y = k\cos(\log x) + k_1\sin(\log x) + \frac{x^2}{5}$

4. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = x^3$

*Ans.* $y = k\cos(2\log x) + k_1\sin(2\log x) + \frac{x^3}{13}$

5. $x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} - 6y = 0$

*Ans.* $y = kx + k_1x^2 + k_2x^3$

## CHAPTER XXVIII

# PRACTICAL APPLICATIONS OF THE CALCULUS

**Discharge of a condenser through a high resistance with no inductance.**

Let K (Fig. 103) denote the capacity of the condenser, E its potential, R the resistance in the circuit and Q the charge; then

$$Q = KE \text{ and } I = \frac{E}{R}$$

$$\frac{dQ}{dt} = -I = K\frac{dE}{dt} = -\frac{E}{R}$$

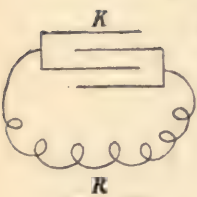

FIG. 103.

therefore $\dfrac{dE}{E} = -\dfrac{dt}{KR}$

On integrating we have

$$\log E = -\frac{t}{KR} + \text{a constant}$$

Let $E = E_0$ when $t = 0$

$$\therefore \log E_0 = \text{the constant}$$

$$\therefore \log E = -\frac{t}{KR} + \log E_0$$

from which we obtain

$$E = E_0 e^{-\frac{t}{KR}} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If $t$, K, R and $E_0$ are known (1) will give the potential

at any instant after the circuit is closed. If R, $t$, E and $E_0$ be known then (1) may be written

$$K = \frac{t}{R \log \frac{E_0}{E}}$$

This determines the capacity.

**To find the law for the growth of a current in a circuit when the voltage suddenly changes from zero to E, taking into account the self-induction in the circuit.**

Let R denote the resistance in the circuit, L the self-induction, I the current, and $t$ the time in seconds. Then

$$E = RI + L\frac{dI}{dt}$$

This equation may be written thus

$$\frac{dI}{\frac{E}{R} - I} = \frac{Rdt}{L}$$

and on integrating we have

$$\log\left(\frac{E}{R} - I\right) = -\frac{Rt}{L} + k$$

where $k$ is a constant.

Let I = 0 when $t$ = 0 therefore

$$k = \log\frac{E}{R}$$

Hence $$\log\left(\frac{E}{R} - I\right) - \log\frac{E}{R} = -\frac{Rt}{L}$$

or $$\log\left(I - \frac{RI}{E}\right) = -\frac{Rt}{L}$$

therefore $$I - \frac{RI}{E} = e^{-\frac{Rt}{L}}$$

therefore $$I = \frac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$$

Helmholtz's equation.

This law will give the current at any time, $t$ seconds after the voltage suddenly changes from zero to V.

If $t = \infty$ the law becomes

$$I = \frac{E}{R} \text{ Ohm's law,}$$

since $e^{-\frac{Rt}{L}} = 0$ when $t = \infty$

**To destroy the effect of self-induction in a circuit by the introduction of a condenser shunt.**

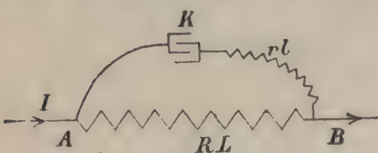


FIG. 104.

Suppose we have a portion of a line AB (Fig. 104) to signal through, and we want the whole arrangement, from A to B, to act as a mere resistance. Let E denote the potential difference between A and B let R denote the resistance in the main circuit AB and L the self-induction. Let $l$ be the self-induction and $r$ the resistance in the condenser shunt.

The total current

$$I = \frac{E}{\text{combined resistance}}$$

The reciprocal of the combined resistance

$$= \frac{1}{R + LD} + \frac{1}{r + lD + \frac{1}{KD}} \text{ where } D = \frac{d}{dt}$$

and K is the capacity of the condenser.

$$\therefore I = \left(\frac{1}{R + LD} + \frac{1}{r + lD + \frac{1}{KD}}\right)E$$

that is,

$$I = \frac{1 + (rK + RK)D + (lK + LK)D^2}{R + (L + RrK)D + (RlK + LrK)D^2 + LlKD^3}E$$

This operation performed on the voltage gives the current. But we want Ohm's Law, viz. $I = \frac{E}{R}$. We want no self-induction or capacity effects, therefore the coefficient of E must be equal to $\frac{1}{R}$

Equate $\frac{1}{R}$ and the coefficient, therefore

$$\frac{1}{R} = \frac{1 + K(r + R)D + K(l + L)D^2}{R + (L + RrK)D + K(Rl + rL)D^2 + LlKD^3}$$

Clearing of fractions and equating the coefficients of like powers of D we have

$$\begin{aligned} RK(r + R) &= L + RrK \quad . \quad . \quad . \quad . \quad (a) \\ RK(l + L) &= K(Rl + Lr) \quad . \quad . \quad . \quad (b) \\ \text{and} \qquad LlK &= 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c) \end{aligned}$$

Consider ($c$) L and K are not 0 therefore $l = 0$ that is, there must be no self-induction in the condenser branch.

Again, if $l = 0$ from equation ($b$) we get $R = r$ and from equation ($a$) we get

$$K = \frac{L}{R^2}$$

That is, the resistance in the main branch must be equal to the resistance in the condenser branch, and the capacity of the condenser must be equal to the self-induction in the main branch divided by the square of its resistance.

Suppose the voltage obeys the law

$$E = E_0 \sin pt$$

we have

$$I = \frac{1+K(R+r)D+K(l+L)D^2}{R+(L+RKr)D+K(Rl+rL)D^2+LlKD^3}E_0 \sin pt$$

$$= \frac{\{1-p^2(L+l)K\}+K(R+r)D}{\{R-p^2(Rl+Lr)K\}+\{(L+RKr)-p^2Ll K\}D}E_0 \sin pt$$

$$= \frac{m+nD}{m'+n'D}E_0 \sin pt \text{ say}$$

$$\therefore I = \sqrt{\frac{m^2+p^2n^2}{m'^2+p^2n'^2}}E_0 \sin\left\{pt+\tan^{-1}\left(\frac{pn}{m}\right)-\tan^{-1}\left(\frac{pn'}{m'}\right)\right\}$$

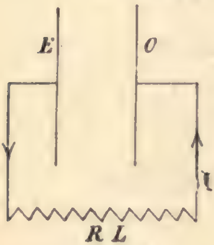

FIG. 105.

See Example page 282.

**To find the law for the discharge of a condenser of capacity of K through a resistance R with self-induction L, in the circuit.**

Let E be the potential difference, R the resistance, L the self-induction, and I the current.

We have $$E = RI + L\frac{dI}{dt} \quad . \quad . \quad . \quad . \quad (a)$$

Also $$Q = KE$$

Q being the quantity of electricity.

Now $$I = -\frac{dQ}{dt} = -K\frac{dE}{dt}$$

therefore $$\frac{dI}{dt} = -K\frac{d^2E}{dt^2}$$

Substituting in (*a*) we get

$$E = -RK\frac{dE}{dt} - LK\frac{d^2E}{dt^2}$$

therefore $$\frac{d^2E}{dt^2}+\frac{R}{L}\frac{dE}{dt}+\frac{1}{LK}E = 0 \quad . \quad . \quad . \quad (b)$$

Assume $$E = ke^{xt}$$

$$\therefore \frac{dE}{dt} = kxe^{xt}$$

and $$\frac{d^2E}{dt^2} = kx^2e^{xt}$$

therefore substituting in ($b$) we get

$$x^2 + \frac{R}{L}x + \frac{1}{LK} = 0$$

the auxiliary equation.

$$\therefore x = -\frac{R}{2L} \pm \sqrt{\left(\frac{R^2}{4L^2} - \frac{1}{LK}\right)}$$

that is,

$$x = -\frac{R}{2L} \pm \sqrt{\left(\frac{1}{LK} - \frac{R^2}{4L^2}\right)}\sqrt{-1} = -a \pm \beta\sqrt{-1}$$

where $a = \dfrac{R}{2L}$ and $\beta = \sqrt{\left(\dfrac{1}{LK} - \dfrac{R^2}{4L^2}\right)}$

If $\dfrac{1}{LK} - \dfrac{R^2}{4L^2}$ be positive the solution of ($b$) is

$$\begin{aligned} E &= e^{-at}(A_1 \cos \beta t + A_2 \sin \beta t) \\ &= e^{-at}C \sin (\beta t + \theta) \quad . \quad . \quad . \quad . \quad . \quad (c) \end{aligned}$$

where $C = \sqrt{A_1{}^2 + B_2{}^2}$ and $\theta = \tan^{-1}\dfrac{A_1}{A_2}$

This shows that if $\dfrac{R}{2L}$ be less than $\dfrac{1}{\sqrt{KL}}$ there will be electrical oscillations, and

$$\sqrt{\frac{1}{LK} - \frac{R^2}{4L^2}} = 2\pi \text{ frequency.}$$

If R be so small that it can be neglected then

$$\frac{2\pi}{T} = \frac{1}{\sqrt{LK}}$$

that is the periodic time of the electrical oscillations is

$$T = 2\pi\sqrt{LK}$$

the frequency being $\dfrac{1}{2\pi\sqrt{LK}}$

Given $K = \frac{1}{4 \times 10^6}$ and $L = 9$ the frequency is

$$\frac{1}{6{\cdot}28}\sqrt{\frac{4 \times 10^6}{9}} = \frac{2000}{18{\cdot}84} = 106{\cdot}1 \text{ oscillations per sec.}$$

If $\frac{1}{LK} - \frac{R^2}{4L^2}$ be zero or negative the roots of the auxiliary equation will be equal or real respectively and there will be no oscillations. That is, the condenser will gradually discharge.

An alternating E.M.F. $E = E_0 \sin pt$ is impressed on two circuits in parallel (Fig. 106) with resistance, self-

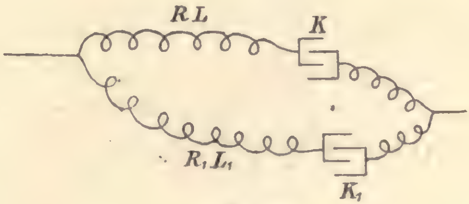


FIG. 106.

induction and capacity in series, it is required to find the law of the current

The current in the top branch is

$$\frac{E}{R + LD + \frac{1}{KD}}$$

and that through the lower branch is

$$\frac{E}{R_1 + L_1 D + \frac{1}{K_1 D}} \quad \text{where } D = \frac{d}{dt}$$

The total current is the sum of these two

$$\therefore \mathrm{I} = \left(\frac{1}{\mathrm{R}+\mathrm{LD}+\frac{1}{\mathrm{KD}}}+\frac{1}{\mathrm{R}_1+\mathrm{L}_1\mathrm{D}+\frac{1}{\mathrm{K}_1\mathrm{D}}}\right)\mathrm{E}_0 \sin pt$$

$$\therefore \mathrm{I} = \left(\frac{a\mathrm{D}+b\mathrm{D}^2+c\mathrm{D}^3}{1+a_1\mathrm{D}+b_1\mathrm{D}^2+c_1\mathrm{D}^3+e\mathrm{D}^4}\right)\mathrm{E}_0 \sin pt$$

$$= \frac{\alpha+\beta\mathrm{D}}{\alpha_1+\beta_1\mathrm{D}}\mathrm{V}_0 \sin pt$$

$$= \sqrt{\frac{\alpha^2+(\beta p)^2}{\alpha_1{}^2+(\beta_1 p)^2}}\sin(pt+\theta-\phi)$$

where $\alpha = -bp^2$, $\beta = a-cp^2$,

$\alpha_1 = 1-b_1p^2+ep^4$ and $\beta_1 = a_1-c_1p^2$

$$\tan\theta = \frac{\beta p}{\alpha} \text{ and } \tan\phi = \frac{\beta_1 p}{\alpha_1}$$

**Euler's theory of struts.**

Let the length of the strut (Fig. 107) be at least thirty times the least diameter, and let it be loaded with a load W producing a deflection $y$ at a distance $x$ from A the origin. The bending moment due to the load is

$$\mathrm{W}y$$

and the resisting moment is $\dfrac{\mathrm{EI}}{\mathrm{R}} = \mathrm{EI}\dfrac{d^2y}{dx^2}$

where E is the modulus of elasticity of the material and I is the least moment of inertia of the cross-section,

FIG. 107.

therefore $$\mathrm{EI}\frac{d^2y}{dx^2} = -\mathrm{W}y \quad . \quad . \quad . \quad . \quad (1)$$

the negative sign being taken since $\dfrac{d^2y}{dx^2}$ is negative since the pillar is concave towards the axis of X when $y$ is positive

Equation (1) may be written

$$\frac{d^2y}{dx^2}+\frac{W}{EI}y = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let $y = ke^{mx}$ and we have

$$\frac{d^2y}{dx^2} = km^2e^{mx}$$

Substituting in (2) we have

$$ke^{mx}\left\{m^2+\frac{W}{EI}\right\} = 0$$

$$\therefore m = \pm\sqrt{\frac{W}{EI}}\sqrt{-1}$$

$$\therefore y = ke^{\sqrt{\frac{W}{EI}}ix}+k_1e^{-\sqrt{\frac{W}{EI}}ix} \text{ where } i = \sqrt{-1}$$

$$\therefore y = A\cos\sqrt{\frac{W}{EI}}x+B\sin\sqrt{\frac{W}{EI}}x$$

To get A and B we have $x = 0$ where $y = 0$

therefore $A = 0$

$$\therefore y = B\sin\sqrt{\frac{W}{EI}}x \quad . \quad . \quad . \quad (3)$$

This result shows that if the pillar be of uniform section and homogeneous, it will assume a curve of sines when loaded axially, B being the maximum value of $y$ From (3) we have, by differentiating,

$$\frac{dy}{dx} = B\sqrt{\frac{W}{EI}}\cos\sqrt{\frac{W}{EI}}x$$

and where $x = \frac{l}{2}, \frac{dy}{dx} = 0$

therefore $\sqrt{\frac{W}{EI}}\frac{l}{2} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ or $\frac{5\pi}{2}$ etc.

$$\text{Assuming} \quad \sqrt{\frac{W}{EI}}\frac{l}{2} = \frac{\pi}{2}$$

$$\text{we obtain} \quad W = \frac{\pi^2 EI}{l^2}$$

where W is the breaking load, when the strut is hinged at the ends. The crushing stress does not appear in this formula so that it breaks by bending only when the length is great compared with the least diameter.

If one end be fixed (Fig. 108) and the other end free to move laterally, then if the upper end be initially at B and when loaded it moves to C a distance $h$ then the bending moment at D is

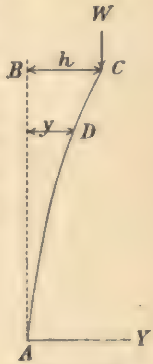

FIG. 108.

$$W(h-y) = EI\frac{d^2y}{dx^2}$$

$$\therefore \frac{d^2y}{dx^2} + \frac{W}{EI}y = \frac{Wh}{EI}$$

The complete solution to this equation is

$$y = h + A\cos\sqrt{\frac{W}{EI}}x + B\sin\sqrt{\frac{W}{EI}}x \quad . \quad (1)$$

$$\text{when } x = 0, \; y = 0 \qquad \therefore h + A = 0$$

$$\text{and } \frac{dy}{dx} = \sqrt{\frac{W}{EI}}\left\{-A\sin\sqrt{\frac{W}{EI}}x + B\cos\sqrt{\frac{W}{EI}}x\right\}$$

$$\text{Where } x = 0, \; \frac{dy}{dx} = 0 \qquad \therefore B = 0$$

$$\therefore y = h - h\cos\sqrt{\frac{W}{EI}}x$$

$$\text{Where } x = l, \; y = h$$

$$\therefore h = h - h\cos\sqrt{\frac{W}{EI}}l$$

$$\therefore \sqrt{\frac{W}{EI}}l = \frac{\pi}{2}$$

$$\therefore W = \frac{\pi^2 EI}{4l^2}$$

If both ends be rigidly fixed in position and direction (Fig. 109) the equation to the shape of the strut is easily shown to be

Fig. 109.

$$y = A\cos\sqrt{\frac{W}{EI}}x$$

and $$\frac{dy}{dx} = -A\sqrt{\frac{W}{EI}}\sin\sqrt{\frac{W}{EI}}x$$

and $$\frac{dy}{dx} = 0 \text{ where } x = \frac{l}{2}$$

$$\therefore \sqrt{\frac{W}{EI}}\frac{l}{2} = \pi$$

$$\therefore W = \frac{4\pi^2 EI}{l^2}.$$

**Strength rule for solid and hollow round shafts.** Let $d$ (Fig. 110) denote the diameter of a solid shaft subjected to a twisting moment, then assuming that the shear stress is proportional to the distance from the centre, the shear stress at a distance $x$ from the centre is

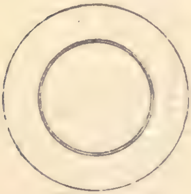

Fig. 110.

$$\frac{2qx}{d}$$

where $q$ is the shear stress at the surface.

The shear force in a concentric ring the breadth of which is $dx$ and radius $x$ is

$$\frac{4\pi q x^2 dx}{d}$$

and the moment of this shear force about the centre is

$$\frac{4\pi q x^3 dx}{d}$$

The torque, T, in the shaft is therefore given by

$$T = \frac{4\pi q}{d}\int_0^{\frac{d}{2}} x^3 dx = \frac{4\pi q}{d}\left[\frac{x^4}{4}\right]_0^{\frac{d}{2}}$$

$$\therefore T = \frac{\pi d^3 q}{16}$$

If $d$ be in inches and $q$ in tons per square inch the torque T will be in tons-inch units.

For a hollow shaft of external radius R and internal radius $r$

$$\begin{aligned} T &= \frac{4\pi q}{d}\int_r^R x^3 dx \\ &= \frac{4\pi q}{d}\left[\frac{x^4}{4}\right]_r^R \\ &= \frac{\pi q}{d}[R^4 - r^4] \\ &= \frac{\pi(d^4 - d_1{}^4)q}{16d} \end{aligned}$$

where $d_1$ is the internal, and $d$ is the external diameter.

**Vibrations natural and forced.**

A mass of M lbs., suspended by a spiral spring which elongates one foot when pulled with a force of $\mu$ poundals, will continue to move up and down with a simple harmonic motion, with undiminishing amplitude, provided there be no frictional resistance to the body's motion.

When a spiral spring is subjected to a pull or push along its axis, it is found that the displacement is proportional to the force exerted, a condition necessary to produce simple harmonic motion.

When the mass M is displaced from its mean position a distance of $x$ feet, the restoring force is therefore $\mu x$ poundals. Using the fundamental equation

Fig. 111.

Force (poundals) = acceleration $\times$ mass (lbs.),

we have $$M\frac{d^2x}{dt^2} = -\mu x \quad . \quad . \quad . \quad . \quad (1)$$

since the expression for an acceleration is $\dfrac{d^2x}{dt^2}$, where $x$ is the displacement and $t$ is the time in seconds reckoned from some instant. The negative sign on the right-hand side of (1) is accounted for by the fact that the restoring force acts in such a way as always to bring the body to its mean position.

From (1) we have, by dividing by M and transposing,

$$\frac{d^2x}{dt^2} + \frac{\mu}{M}x = 0 \quad . \quad . \quad . \quad . \quad (2)$$

The solution of (2) is got by assuming $x = ke^{mt}$, and forming the auxiliary equation, the roots of which are

$$m = \sqrt{\frac{\mu}{M}}\sqrt{-1}, \text{ or, } -\sqrt{\frac{\mu}{M}}\sqrt{-1}$$

consequently the general solution to (2) is

$$x = ke^{\sqrt{\frac{\mu}{M}}\sqrt{-1}t} + k_1e^{\sqrt{\frac{\mu}{M}}\sqrt{-1}t} \quad . \quad . \quad (3)$$

Now $$e^{a\sqrt{-1}t} = \cos at + \sqrt{-1}\sin at$$

and $$e^{-a\sqrt{-1}t} = \cos at - \sqrt{-1}\sin at$$

therefore the solution of (3) is

$$x = A\cos\sqrt{\frac{\mu}{M}}t + B\sin\sqrt{\frac{\mu}{M}}t \quad . \quad . \quad (4)$$

If we reckon time from the instant the body is passing its mean position, we have

$$x = \mathrm{B} \sin \sqrt{\frac{\mu}{\mathrm{M}}}t \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If we reckon time from the instant the body is at rest, we have from (4), by differentiating—

$$\frac{dx}{dt} = -\mathrm{A}\sqrt{\frac{\mu}{\mathrm{M}}} \sin \sqrt{\frac{\mu}{\mathrm{M}}}t + \mathrm{B}\sqrt{\frac{\mu}{\mathrm{M}}} \cos \sqrt{\frac{\mu}{\mathrm{M}}}t$$

and $\frac{dx}{dt} = 0$ when $t = 0$

$$\therefore \mathrm{B} = 0$$

Therefore (4) becomes

$$x = \mathrm{A} \cos \sqrt{\frac{\mu}{\mathrm{M}}}t$$

Equation (4) can be expressed in the form

$$x = \mathrm{C} \sin \left(\sqrt{\frac{\mu}{\mathrm{M}}}t + \theta\right) \quad . \quad . \quad . \quad . \quad (6)$$

where $\mathrm{C} = \sqrt{\mathrm{A}^2 + \mathrm{B}^2}$ and $\tan \theta = \frac{\mathrm{A}}{\mathrm{B}}$

and (6) is the equation to the motion when we reckon time from the instant the body is $\mathrm{C} \sin \theta$ from its mean position.

To find the periodic time of a complete vibration we have

$$\frac{2\pi}{\mathrm{T}} = \sqrt{\frac{\mu}{\mathrm{M}}}$$

Let $\mathrm{M} = 40$ lbs., and let $\mu = 60$ poundals, then

$$\mathrm{T} = 2 \times 3{\cdot}14\sqrt{\tfrac{40}{60}} = 2 \times 3{\cdot}14 \times {\cdot}816 = 5{\cdot}12 \text{ secs.}$$

Again suppose the mass to vibrate in a resisting medium, the frictional resistance of which is $2f$ poundals when the velocity is one foot per second, and suppose the frictional

resistance to be proportional to the velocity, then the equation of motion is

$$M\frac{d^2x}{dt^2} + 2f\frac{dx}{dt} + \mu x = 0 \quad . \quad . \quad . \quad . \quad (7)$$

The roots of the auxiliary equation of (7) are

$$m = -\frac{f}{M} \pm \sqrt{\frac{f^2}{M^2} - \frac{\mu}{M}}$$

Suppose $\frac{f^2}{M^2} < \frac{\mu}{M}$, then the roots are imaginary and

$$m = -\frac{f}{M} \pm \sqrt{\frac{\mu}{M} - \frac{f^2}{M^2}}\sqrt{-1}$$

$$= -a \pm \beta i$$

where $a = \frac{f}{M}$, $\beta = \sqrt{\frac{\mu}{M} - \frac{f^2}{M^2}}$, $i = \sqrt{-1}$

The solution of (7) is

$$x = e^{-at}\{A \cos \beta t + \beta \sin \beta t\}$$

or $$x = e^{-at} C \sin (\beta t + \theta) \quad . \quad . \quad . \quad . \quad . \quad (8)$$

On plotting $x$ vertically and $t$ horizontally we get a curve as shown in Fig. 112.

If we take the origin at the point where the curve cuts the time axis, equation (8) becomes

$$x = Ce^{-at} \sin \beta t \quad . \quad . \quad . \quad . \quad (9)$$

In a particular case the periodic time T was observed to be 1·256 seconds, and the ratio of $x_2$ to $x_1$, $x_3$ to $x_2$, etc. was ·284; $\mu$ was 116 poundals and $x_1$ was ·1 foot. The mass M frictional resistance $2f$ and C were found in the following way.

Equation (9) gives the displacement $x_1$ at any instant $t_1$ seconds. The displacement $x_2$ at time $t_1 + \delta t$ is

$$x_2 = Ce^{-a(t_1+\delta t)} \sin \beta(t_1 + \delta t)$$

that is, $$x_2 = e^{-a\delta t}\, Ce^{-at_1} \sin \beta(t_1 + \delta t)$$

Similarly, $$x_3 = e^{-a\delta t}\, Ce^{-at_1} \sin \beta(t_1 + 2\delta t)$$

Let $\delta t$ have such a value that

$$\beta\delta t = \pi$$

therefore $\sin \beta t_1 = \sin \beta(t_1 + 2\delta t)$

therefore $\dfrac{x_3}{x_1} = e^{-2a\delta t} = (\cdot 284)^2$

that is, $-2a\delta t = 2\cdot 302 \times 2 \log_{10} \cdot 284$

whence $a\delta t = 1\cdot 258.$

Again $\delta t = \frac{1}{2} \times \text{periodic time} = \cdot 628$

and $\beta\delta t = \pi, \quad \therefore \beta = \dfrac{\pi}{\cdot 628} = 5.$

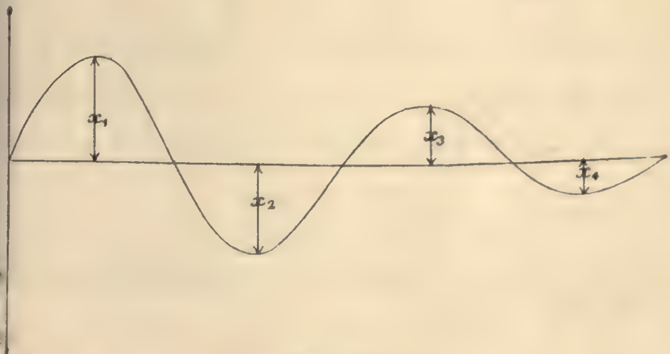


FIG. 112.

Also $a\delta t$ which is called the *logarithmic decrement*, is 1·258

$$\therefore a = \frac{1\cdot 258}{\cdot 628} = 2 \text{ nearly}$$

Again $\beta^2 = \dfrac{\mu}{M} - \dfrac{f^2}{M^2}$ and $a = \dfrac{f}{M} = 2$

$$\therefore 25 = \frac{116}{M} - 4$$

$$\therefore M = 4 \text{ lbs.}$$

and $2f = 4M = 16$ poundals.

To get C, $x_1$ was ·1 foot, and the corresponding value

of $t_1$ measured from the origin was one-fourth the periodic time $=$ ·314 sec.

On substituting in (9) we have

$$\cdot 1 = e^{-2 \times \cdot 314}\text{C} \sin \frac{\pi}{2}$$

from which $\text{C} = \cdot 187$ foot, the maximum displacement, if there were no friction

$$\therefore x = e^{-2t} \times \cdot 187 \sin 5t$$

which gives the displacement of the body at any instant.

Suppose $f$ to gradually increase, consequently $\beta$ will get smaller and smaller, but $\beta = \frac{2\pi}{\text{T}}$ where T is the periodic time, therefore T will get greater and greater as $f$ increases and when

$$\frac{f^2}{\text{M}^2} = \frac{\mu}{\text{M}}$$

that is, when the two roots of the auxiliary equation are equal, the solution is

$$x = (\text{A} + \text{B}t)e^{-at} \quad . \quad . \quad . \quad . \quad (10)$$

where A and B are constants depending upon given data. From (10) it is obvious that, as time goes on, $x$ diminishes, but can never become negative, if A and B be positive; that is, the body approaches nearer and nearer to the position which it would occupy if at rest if there were no friction, but never reaches it.

The foregoing solutions apply equally as well to electric oscillations.

**Forced vibrations.**—In the foregoing investigations the upper end of the spiral spring was fixed. Now suppose the upper end of the spring to get a simple harmonic motion, whose amplitude is $a$ and whose periodic time is $\text{T}_1$ and let

us assume that there is no frictional resistance, the equation of motion is

$$M\frac{d^2x}{dt^2} + \mu(x - y) = 0 \quad . \quad . \quad . \quad (12)$$

where $x$ is the displacement of the body from its mean position, and $y$ is the displacement of the upper end of the spiral spring at any instant.

But $y = a \sin pt$, therefore (12) becomes

$$M\frac{d^2x}{dt^2} + \mu x = \mu a \sin pt \quad . \quad . \quad . \quad (13)$$

where $p = \dfrac{2\pi}{T_1}$

On dividing (13) by M and substituting $n^2$ for $\dfrac{\mu}{M}$ and $m$ for $\dfrac{\mu a}{M}$ we have

$$\frac{d^2x}{dt^2} + n^2x = m \sin pt \quad . \quad . \quad . \quad (14)$$

A solution of this equation is

$$x = A \sin pt + B \cos pt \quad . \quad . \quad . \quad (15)$$

Differentiating (15) we have

$$\frac{d^2x}{dt^2} = -p^2A \sin pt - Bp^2 \cos pt$$

Substituting in (14) we get

$$-p^2A \sin pt - Bp^2 \cos pt + n^2A \sin pt + n^2B \cos pt = m \sin pt$$

Equating the coefficients of $\sin pt$, and also of $\cos pt$ on both sides, we have

$$-p^2A + n^2A = m, \text{ and } -Bp^2 + n^2B = 0$$

whence $\qquad A = \dfrac{m}{n^2 - p^2}$ and $B = 0$

therefore (15) becomes

$$x = \frac{m}{n^2 - p^2} \sin pt \quad . \quad . \quad . \quad . \quad (16)$$

Note that the amplitude of the motion of M increases as the value of $p$ approaches that of $n$ and when $p$ is equal to $n$ the amplitude is infinity.

**Relation between the periodic times of a damped and undamped vibration.**

The general equation of the motion of a body vibrating without friction is

$$\frac{d^2x}{dt^2} + \frac{\mu}{M}x = 0$$

A particular solution of which is

$$x = A \sin \sqrt{\frac{\mu}{M}}t$$

where $$\sqrt{\frac{\mu}{M}} = \frac{2\pi}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and T is the periodic time.

The equation of motion of the same body when friction is taken into account is

$$\frac{d^2x}{dt^2} + \frac{2f}{M}\frac{dx}{dt} + \frac{\mu}{M}x = 0$$

the solution of which is

$$x = Ce^{-\frac{ft}{M}} \sin \sqrt{\frac{\mu}{M} - \frac{f^2}{M^2}}t$$

therefore $$\sqrt{\frac{\mu}{M} - \frac{f^2}{M^2}} = \frac{2\pi}{T_1} \quad . \quad . \quad . \quad . \quad (2)$$

where $T_1$ is the periodic time with friction.

From (1) $$\sqrt{\frac{\mu}{M}} = \frac{2\pi}{T}$$

where T is the periodic time without friction,

$$\therefore \frac{\mu}{M} = \frac{4\pi^2}{T^2} = \frac{4\pi^2}{T_1^2} + \frac{f^2}{M^2}$$

$$\therefore \frac{1}{T^2} = \frac{1}{T_1^2} + \frac{f^2}{4\pi^2 M^2}$$

This shows that the periodic time increases as the friction increases.

Let T = 2, $f$ = 40 poundals, M = 20 lbs., then

$$\frac{1}{T_1^2} = \frac{1}{4} - \frac{1600}{4 \times 9{\cdot}86 \times 400}$$

$$\therefore T_1 = 2{\cdot}5 \text{ seconds nearly}$$

**Flow of heat through a medium.**

Consider a small cubical block of the medium, the edges

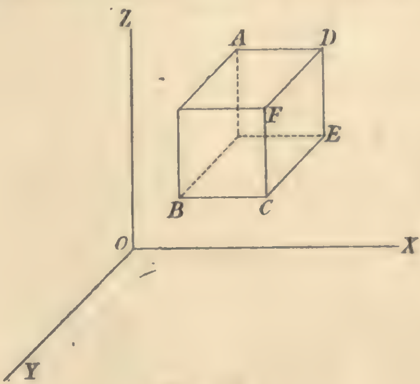


FIG. 113.

in the directions of the axes of X, Y and Z (Fig. 113) being respectively $dx$, $dy$ and $dz$

The flow into the block through the face AB is

$$-k\frac{dv}{dx}dydz$$

where $\frac{dv}{dx}$ is the temperature gradient, and $k$ the conductivity

The flow through unit area per second is a function of the distance of the area from the plane YZ therefore

$$-k\frac{dv}{dx} = f(x)$$

The flow through unit area per second at distance $x + dx$ is $f(x + dx)$ but

$$f(x + dx) = f(x) + dx\frac{df}{dx} + \ldots$$

$$\therefore f(x + dx) - f(x) = + dx\frac{df}{dx}$$

where $dx$ is very small

Now $f(x)$ represents the heat flowing in through unit area at AB and $f(x + dx)$ represents the heat flowing out through unit area at FE, the heat stopping in block of unit area and thickness $dx$ is

$$f(x) - f(x + dx) = -dx\frac{df}{dx}$$

Now $f(x)$ is $-k\frac{dv}{dx}$ and $f(x) - f(x + dx) = k\frac{d^2v}{dx^2}dx$

The heat stopping in the cube the edges of which are $dx$, $dy$, $dz$, is

$$k\frac{d^2v}{dx^2}dxdydz$$

and this heat goes to raise the temperature of the block. The heat stopping in per second is

$=$ vol. $\times$ rate of increase of temp. $\times$ specific heat $\times$ density

$= dxdydz\frac{dv}{dt}s\rho$

where $s$ is the specific heat, and $\rho$ the density

$$\therefore kdxdydz\frac{d^2v}{dx^2} = dxdydz\rho s\frac{dv}{dt}$$

$$\therefore \frac{d^2v}{dx^2} = \frac{\rho s}{k}\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

This is the differential equation for the linear flow of heat in a given direction.

Similarly if we consider the heat flowing in, in the direction of the axis of Y we have

$$\frac{d^2v}{dy^2} = \frac{\rho s}{k}\frac{dv}{dt}$$

and in the direction of $z$ we have

$$\frac{d^2v}{dz^2} = \frac{\rho s}{k}\frac{dv}{dt}$$

Therefore the total heat stopping in to raise the temperature is

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2} = \frac{\rho s}{k}\frac{dv}{dt} \quad . \quad . \quad . \quad (2)$$

If the temperature of the medium be in a state of equilibrium, then $\frac{dv}{dt} = 0$ and (2) assumes the form

$$\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2} = 0 \quad . \quad . \quad . \quad (3)$$

which is sometimes written thus

$$\nabla^2 v = 0$$

This is called **Laplace's equation** and is of very great importance in its application to Mechanical and Electrical problems.

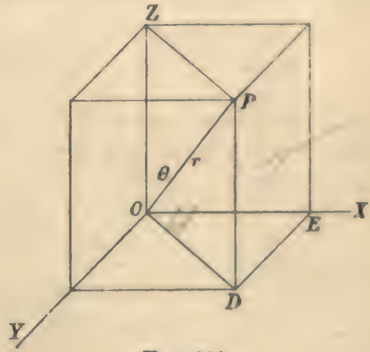


FIG. 114.

Let $x$, $y$, $z$ (Fig. 114) be the polar coordinates of a point P, OP $= r$. Let the angle which OP makes with the axis

of $z$ be $\theta$ and let the projection of OP on the plane XY make an angle $\phi$ with the axis of X then

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned}$$

On transforming (3) to polar coordinates, we have

$$\frac{d^2v}{dr^2} + \frac{1}{r^2}\frac{d^2v}{d\theta^2} + \frac{2}{r}\frac{dv}{dr} + \frac{1}{r^2\sin^2\theta}\frac{d^2v}{d\phi^2} + \frac{\cot\theta}{r^2}\frac{dv}{d\theta} = 0 \quad (4)$$

(*See* Todhunter's *Differential Calculus*, page 187.)

If $v$ be independent of $\phi$, (4) reduces to

$$\frac{d^2v}{dr^2} + \frac{1}{r^2}\frac{d^2v}{d\theta^2} + \frac{2}{r}\frac{dv}{dr} + \frac{\cot\theta}{r^2}\frac{dv}{d\theta} = 0$$

which may be written in the form

$$\frac{1}{r^2}\left\{\frac{d}{dr}\left(r^2\frac{dv}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dv}{d\theta}\right)\right\} = 0\ . \quad (5)$$

Let $v = \mathrm{RP}$ where R is a function of $r$ and P is a function of $\theta$ On substituting in (5) we have

$$\frac{r^2}{\mathrm{R}}\frac{d^2\mathrm{R}}{dr^2} + \frac{2r}{\mathrm{R}}\frac{d\mathrm{R}}{dr} = -\left(\frac{1}{\mathrm{P}}\frac{d^2\mathrm{P}}{d\theta^2} + \frac{\cot\theta}{\mathrm{P}}\frac{d\mathrm{P}}{d\theta}\right) \ . \quad (6)$$

The left-hand side is a function of $r$ only, and the right-hand side is a function of $\theta$ only, therefore each side is equal to a constant.

$$\therefore \frac{r^2}{\mathrm{R}}\frac{d^2\mathrm{R}}{dr^2} + \frac{2r}{\mathrm{R}}\frac{d\mathrm{R}}{dr} = k \quad . \ . \ . \quad (7)$$

Let $\mathrm{R} = r^m$ where $m$ is an integer

$$\therefore \frac{r^2}{\mathrm{R}}m(m-1)r^{m-2} + \frac{2r}{\mathrm{R}}mr^{m-1} = k$$

$$\therefore m(m+1) = k$$

$$\therefore\ m(m+1)+\frac{1}{\mathrm{P}}\frac{d^2\mathrm{P}}{d\theta^2}+\frac{\cot\theta}{\mathrm{P}}\frac{d\mathrm{P}}{d\theta} = 0$$

$$\therefore\ \frac{d^2\mathrm{P}}{d\theta^2}+\cot\theta\frac{d\mathrm{P}}{d\theta}+m(m+1)\mathrm{P} = 0 \quad . \quad (8)$$

If $\mu = \cos\theta$ then

$$\frac{d^2\mathrm{P}}{d\theta^2} = (1-\mu^2)\frac{d^2\mathrm{P}}{d\mu^2}-\mu\frac{d\mathrm{P}}{d\mu}$$

$$\text{and } \frac{d\mathrm{P}}{d\theta} = -\sqrt{1-\mu^2}\frac{d\mathrm{P}}{d\mu}$$

On substituting in (8) we have

$$(1-\mu^2)\frac{d^2\mathrm{P}}{d\mu^2}-2\mu\frac{d\mathrm{P}}{d\mu}+m(m+1)\mathrm{P} = 0 \quad . \quad (9)$$

This is Legendre's equation which is very important for solving problems on potential, flow of heat, etc.

A function P which satisfies (8) is called a **surface zonal harmonic.**

Substituting $m(m+1)$ for $k$ in (7) we have

$$r^2\frac{d^2\mathrm{R}}{dr^2}+2r\frac{d\mathrm{R}}{dr}-m(m+1)\mathrm{R} = 0$$

Writing D for $r\dfrac{d}{dr}$ we have

$$\{\mathrm{D}(\mathrm{D}-1)+2\mathrm{D}-m(m+1)\}\mathrm{R} = 0$$

$$\therefore\ \{\mathrm{D}^2+\mathrm{D}-m(m+1)\}\mathrm{R} = 0$$

From which we obtain

$$\mathrm{R} = kr^m+k_1r^{-(m+1)}$$

For the complete solution of Legendre's equation *see* Forsyth's *Differential Equations*, page 143. Assuming $m$ in Legendre's equation, to be a positive integer, the solution is

$$\mathrm{P}_m = \frac{\underline{|2m}}{2^m\,\underline{|m}\,\underline{|m}}\Big[\mu^m-\frac{m(m-1)}{2(2m-1)}\mu^{m-2}$$

$$+\frac{m(m-1)(m-2)(m-3)}{2\times4(2m-1)(2m-3)}\mu^{m-4}\ldots\Big]$$

$$
\begin{aligned}
\text{If } m &= 0 \text{ then } P_0 = 1 \\
m &= 1 \quad ,, \quad P_1 = \cos\theta \quad \text{since } \mu = \cos\theta \\
m &= 2 \quad ,, \quad P_2 = \frac{3\cos^2\theta}{2} - \frac{1}{2} \\
m &= 3 \quad ,, \quad P_3 = \frac{5\cos^3\theta}{2} - \frac{3\cos\theta}{2} \\
m &= 4 \quad ,, \quad P_4 = \frac{35\cos^4\theta}{8} - \frac{15}{4}\cos^2\theta + \frac{3}{8} \\
m &= 5 \quad ,, \quad P_5 = \tfrac{63}{8}\cos^5\theta - \tfrac{35}{4}\cos^3\theta + \tfrac{15}{8}\cos\theta \\
m &= 6 \quad ,, \quad P_6 = \tfrac{231}{16}\cos^6\theta - \tfrac{315}{16}\cos^4\theta \\
&\qquad\qquad\qquad + \tfrac{105}{16}\cos^2\theta - \tfrac{5}{16}
\end{aligned}
$$

. . . . . . . . . .

For a table of surface zonal harmonics up to $P_7$ *see* Byerley's *Harmonic Functions*, pages 60, 61.

We assumed

$$v = RP$$

and we have found that

$$R = kr^m + k_1 r^{-(m+1)}$$

and denoting the solution of (9) by

$$P_m(\mu)$$

we have $$v = \{kr^m + k_1 r^{-(m+1)}\} P_m(\mu)$$

which is a solution of (5).

*See* Byerley's *Spherical Harmonics* for practical problems on Potential, Heat conduction, etc.

## CHAPTER XXIX

# DIFFERENTIAL COEFFICIENT OF A FUNCTION OF TWO OR MORE VARIABLES, AND OF IMPLICIT FUNCTIONS

LET $v = f(x, y)$ where $x$ and $y$ are both variables, and suppose $x$ and $y$ to receive finite increments $\delta x$ and $\delta y$ respectively, and let $\delta v$ denote the corresponding increment of $v$ We have

$$v + \delta v = f(x + \delta x, y + \delta y)$$

$$\therefore \delta v = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= f(x + \delta x, y + \delta y) - f(x, y + \delta y) + f(x, y + \delta y) - f(x, y)$$

$$= \frac{\{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\}}{\delta x}\delta x$$

$$+ \frac{\{f(x, y + \delta y) - f(x, y)\}}{\delta y}\delta y$$

Suppose $\delta x$ and $\delta y$ and consequently $\delta v$ to diminish indefinitely; therefore

$$dv = \frac{\{f(x + dx, y + dy) - f(x, y + dy)\}}{dx}dx$$

$$+ \frac{\{f(x, y + dy) - f(x, y)\}}{dy}dy$$

that is, $$dv = \left(\frac{dv}{dx}\right)dx + \left(\frac{dv}{dy}\right)dy \quad . \quad . \quad . \quad . \quad (a)$$

where $$\left(\frac{dv}{dx}\right) = \frac{f(x + dx, y + dy) - f(x, y + dy)}{dx}$$

$$= \frac{f(x + dx, y) - f(xy)}{dx}$$

when $dy$ is indefinitely small. That is, $\left(\dfrac{dv}{dx}\right)$ is the differential coefficient of $f(x, y)$ treating $y$ as a constant, and

$$\left(\frac{dv}{dy}\right) = \frac{f(x, y+dy) - f(x, y)}{dy}$$

that is, the differential coefficient of $f(x, y)$ treating $x$ as a constant.

$\left(\dfrac{dv}{dx}\right)$ and $\left(\dfrac{dv}{dy}\right)$ are called the partial differential coefficients of $f(x, y)$ with respect to $x$ and $y$ respectively, and $dv$ is the complete differential of $f(x, y)$ when $x$ and $y$ both vary. Therefore the complete differential of a function of two variables is the sum of the differentials of the function with respect to the two variables.

Let $\quad v = ax^2 + 2bxy + cy^2$

therefore $\quad dv = 2(ax + by)dx + 2(bx + cy)dy$

Here $\quad \left(\dfrac{dv}{dx}\right)dx = 2(ax + by)dx$

and $\quad \left(\dfrac{dv}{dy}\right)dy = 2(bx + cy)dy$

Let $\quad v = x \sin y$

therefore $\quad dv = \sin y dx + x \cos y dy$

Let $\quad v = x^n y^m$

therefore $\quad dv = nx^{n-1}y^m dx + my^{m-1}x^n dy$

If $v = f(x, y, z)$ and if $x$, $y$ and $z$ receive infinite increments, $\delta x$, $\delta y$ and $\delta z$ and in consequence let $v$ become $v + \delta v$ therefore

$$v + \delta v = f(x + \delta x, y + \delta y, z + \delta z)$$

therefore

$$\begin{aligned}
\delta v &= f(x + \delta x, y + \delta y, z + \delta z) - f(x, y, z) \\
&= f(x + \delta x, y + \delta y, z + \delta z) - f(x, y + \delta y, z + \delta z) \\
&\quad + f(x, y + \delta y, z + \delta z) - f(x, y, z + \delta z) \\
&\quad + f(x, y, z + \delta z) - f(x, y, z)
\end{aligned}$$

$$= \frac{\{f(x+\delta x\,,\, y+\delta y\,,\, z+\delta z) - f(x\,,\, y+\delta y\,,\, z+\delta z)\}}{dx}dx$$

$$+ \frac{\{f(x\,,\, y+\delta y\,,\, z+\delta z) - f(x\,,\, y\,,\, z+\delta z)\}}{dy}dy$$

$$+ \frac{\{f(x\,,\, y\,,\, z+\delta z) - f(x\,,\, y\,,\, z)\}}{dz}dz$$

that is $\quad dv = \left(\frac{dv}{dx}\right)dx + \left(\frac{dv}{dy}\right)dy + \left(\frac{dv}{dz}\right)dz$

when $\delta x$, $\delta y$ and $\delta z$ become indefinitely small.

Similarly, we could extend this process of reasoning to a function of any number of variables.

Let $v = f(x\,,\, y\,,\, z\,,\, .\ .\ .)$ therefore

$$dv = \left(\frac{dv}{dx}\right)dx + \left(\frac{dv}{dy}\right)dy + \left(\frac{dv}{dz}\right)dz\ .\ .\ .$$

Let $\quad v = \frac{x^2}{a^2} + \frac{y^2}{h^2} + \frac{z^2}{c^2}$

$$\therefore dv = \left(\frac{2x}{a^2}\right)dx + \left(\frac{2y}{h^2}\right)dy + \left(\frac{2z}{c^2}\right)dz$$

In the equation ($a$), page 319, if $v = 0$ we have $f(x\,,\, y) = 0$ and we say that $x$ is an implicit function of $y$ or $y$ is an implicit function of $x$ Since $f(x\,,\, y) = 0$ for all values of $x$ and $y$ therefore

$$f(x+\delta x\,,\, y+\delta y) = 0 \text{ hence } dv = 0$$

therefore $\quad \left(\frac{dv}{dx}\right)dx + \left(\frac{dv}{dy}\right)dy = 0$

therefore $\quad \frac{dy}{dx} = -\frac{\left(\frac{dv}{dx}\right)}{\left(\frac{dv}{dy}\right)}$

Find $\frac{dy}{dx}$ given $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

Here $\dfrac{dv}{dx} = \dfrac{2x}{a^2}$ and $\dfrac{dv}{dy} = \dfrac{2y}{b^2}$

therefore $\dfrac{dy}{dx} = -\dfrac{b^2}{a^2}\dfrac{x}{y}$

To find $\dfrac{dv}{dx}$ where $v = y^3 + z^3 + 3ayz$ and $y = \tan x$, $z = a^x$

Here $v = f(y, z)$ $\therefore dv = \left(\dfrac{dv}{dy}\right)dy + \left(\dfrac{dv}{dz}\right)dz$

and since $y$ and $z$ are functions of $x$

$$\frac{dv}{dx} = \left(\frac{dv}{dy}\right)\frac{dy}{dx} + \left(\frac{dv}{dz}\right)\frac{dz}{dx} \quad . \quad . \quad . \quad . \quad (b)$$

Now

$$\left(\frac{dv}{dy}\right) = 3(y^2 + az), \quad \frac{dy}{dx} = \sec^2 x, \quad \left(\frac{dv}{dz}\right) = 3(z^2 + ay)$$

and $$\frac{dz}{dx} = a^x \log a$$

On substituting in $(a)$ we get

$$\frac{dv}{dx} = 3\{(y^2 + az)\sec^2 x + (z^2 + ay)a^x \log a\}$$

$$\therefore \frac{dv}{dx} = 3\{(\tan^2 x + a^{x+1})\sec^2 x + (a^{2x} + a\tan x)a^x \log a\}$$

Let $v = \tan^{-1}\left(\dfrac{z}{y}\right)$ where $z = f(x)$ and $y = \phi(x)$ find $\dfrac{dv}{dx}$

$$\frac{dv}{dz} = \frac{y}{y^2 + z^2} \quad \text{and} \quad \frac{dv}{dy} = -\frac{z}{y^2 + z^2}$$

Substituting these values in $(b)$ we get

$$\frac{dv}{dx} = \frac{1}{y^2 + z^2}\{yf'(x) - z\phi'(x)\}$$

If $v = \dfrac{z^2 - y^2}{z^2 + y^2}$ find $\dfrac{dv}{dx}$ where $y$ and $z$ are functions of $x$

We have $$\frac{dv}{dx} = \left(\frac{dv}{dz}\right)\frac{dz}{dx} + \left(\frac{dv}{dy}\right)\frac{dy}{dx}$$

Here $\dfrac{dv}{dz} = \dfrac{4y^2z}{(z^2+y^2)^2}$ and $\dfrac{dv}{dy} = \dfrac{-4yz^2}{(z^2+y^2)^2}$

$$\therefore \frac{dv}{dx} = \frac{4y^2z}{(z^2+y^2)^2}\frac{dz}{dx} - \frac{4yz^2}{(z^2+y^2)^2}\frac{dy}{dx}$$
$$= \frac{4yz}{(z^2+y^2)^2}\{y\phi'(x) - zf'(x)\}$$

where $y = f(x)$ and $z = \phi(x)$

## CHAPTER XXX

# MAXIMA AND MINIMA FUNCTIONS OF TWO OR THREE VARIABLES

By an extension of Taylor's Theorem it can be proved that

$$f(x+\alpha\,,\,y+\beta)-f(x\,,\,y) \;=\; \alpha\frac{df}{dx}+\beta\frac{df}{dy}$$
$$+\frac{1}{\underline{|2}}\left\{\alpha^2\frac{d^2f}{dx^2}+2\alpha\beta\frac{d^2f}{dxdy}+\beta^2\frac{d^2f}{dy^2}\right\}$$
$$+\frac{1}{\underline{|3}}\left\{\alpha^3\frac{d^3f}{dx^3}+3\alpha^2\beta\frac{d^3f}{dx^2dy}+3\alpha\beta^2\frac{d^3f}{dxdy^2}+\beta^3\frac{d^3f}{dy^3}\right\}+\text{etc.} \quad \text{(A)}$$

Space does not permit us to give the proof of this expansion, and we therefore refer the reader to more advanced treatises on the subject.

Now, $\frac{d^2f}{dxdy}$ means that $f(x\,,\,y)$ is differentiated, first with regard to $y$ treating $x$ as constant, and the result differentiated with regard to $x$ treating $y$ as constant. $\frac{d^3f}{dx^2dy}$ means that $f(x\,,\,y)$ is differentiated first with regard to $y$ treating $x$ as constant, and the result differentiated twice with regard to $x$ treating $y$ as constant, etc. If $f(x\,,\,y)$ be a maximum or a minimum when $x = a$ and $y = b$ therefore it follows that

$$f(a\pm\alpha\,,\,b\pm\beta)-f(a\,,\,b)$$

must be negative or positive respectively; and when $\alpha$ and $\beta$ are very small, the sign of the right-hand side of A is the same as the sign of

$$\alpha\frac{df}{dx}+\beta\frac{df}{dy}$$

therefore $\quad \dfrac{df}{dx} = 0$ and $\dfrac{df}{dy} = 0$

otherwise $\quad f(a + \alpha\,,\ b + \beta) - f(a\,,\ b)$

and $\quad f(a - \alpha\,,\ b - \beta) - f(a\,,\ b)$

would have opposite signs. Again, for a maximum or a minimum

$$\alpha^2\frac{d^2f}{dx^2} + 2\alpha\beta\frac{d^2f}{dxdy} + \beta^2\frac{d^2f}{dy^2} \quad . \quad . \quad . \quad (B)$$

must be negative or positive respectively, if not zero, for very small values of $\alpha$ and $\beta$

Denoting the differential coefficients $\dfrac{d^2f}{dx^2}$, $\dfrac{d^2f}{dxdy}$ and $\dfrac{d^2f}{dy^2}$ by X , Y and Z respectively, we have

$$\alpha^2X + 2\alpha\beta Y + \beta^2Z \text{ instead of (B)}$$

that is, $$\frac{\beta^2}{X}\left\{\left(X\frac{\alpha}{\beta} + Y\right)^2 + XZ - Y^2\right\}$$

In order that this expression may be always positive or negative for small values of $\alpha$ and $\beta$ it is obvious that $XZ - Y^2$ must not be negative; therefore the condition for a real maximum or minimum is that $\dfrac{d^2f}{dx^2} \times \dfrac{d^2f}{dy^2}$ must be greater than or equal to $\left(\dfrac{d^2f}{dxdy}\right)^2$

If this condition be satisfied, $f(x\,,\ y)$ will be a maximum or minimum, according as X , that is, $\dfrac{d^2f}{dx^2}$ is negative or positive; since $\left(X\dfrac{\alpha}{\beta} + Y\right)^2$ is always positive and also $\beta^2$ is positive.

Therefore we must have the following conditions satisfied, if $f(xy)$ be a maximum or minimum.

$$\frac{df}{dx} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{df}{dy} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{d^2f}{dx^2} = \text{negative quantity for a } maximum \quad . \quad (3)$$

$$\frac{d^2f}{dx^2} = \text{positive quantity for a } minimum \quad . \quad (4)$$

and
$$\frac{d^2f}{dx^2} \times \frac{d^2f}{dy^2} - \left(\frac{d^2f}{dxdy}\right)^2$$

must not be a negative quantity. . . . . . . . . (5)

If $\left(\frac{d^2f}{dx^2}\frac{d^2f}{dy^2}\right) - \left(\frac{d^2f}{dxdy}\right)^2$ be negative, then $f(x, y)$ will be neither a maximum nor a minimum.

**To find the maximum or minimum value of** $v = x^2 + y^2 - 4ax - 6by$

Here
$$\frac{dv}{dx} = 2x - 4a = 0$$

and
$$\frac{dv}{dy} = 2y - 6b = 0$$

for a maximum or minimum. Therefore

$$x = 2a, \text{ and } y = 3b$$
$$\therefore v = -(4a^2 + 9b^2)$$

$$\frac{d^2v}{dx^2} = 2 \text{ a positive quantity, and } \frac{d^2v}{dxdy} = 0$$

Therefore $v = -(4a^2 + 9b^2)$ is a minimum value.

**To find the point the sum of the squares of whose distance from two given points $(a_1, b_1)$ and $(a_2, b_2)$ may be the least possible.**

The result is evident, but we give it as an illustration of the rules employed.

Let $(x, y)$ be the required point; then

$$u = (x-a_1)^2 + (y-b_1)^2 + (x-a_2)^2 + (y-b_2)^2$$

is to be a minimum

$$\left.\begin{aligned} \frac{du}{dx} &= 2x - a_1 - a_2 = 0 \\ \text{and} \qquad \frac{du}{dy} &= 2y - b_1 - b_2 = 0 \end{aligned}\right\} \text{for a minimum.}$$

Also $\dfrac{d^2u}{dx^2} = 2$ a positive quantity,

and $\dfrac{d^2u}{dxdy} = 0$

Therefore $u$ is a minimum, and the co-ordinates of the required point are

$$x = \frac{a_1 + a_2}{2} \qquad y = \frac{b_1 + b_2}{2}$$

that is, the middle point of the straight line joining $(a_1, b_1)$ and $(a_2, b_2)$

**Maxima and minima of three variables.**

If $f(x+\alpha, y+\beta, z+\gamma)$ be expanded in powers of $\alpha$, $\beta$ and $\gamma$, by an extension of Taylor's Theorem, we have

$$f(x+\alpha, y+\beta, z+\gamma) - f(x, y, z) = \alpha\frac{df}{dx} + \beta\frac{df}{dy} + \gamma\frac{df}{dy}$$
$$+ \frac{\alpha^2}{2}\frac{d^2f}{dx^2} + \frac{\beta^2}{2}\frac{d^2f}{dy^2} + \frac{\gamma^2}{2}\frac{d^2f}{dz^2} + \alpha\beta\frac{d^2f}{dxdy} + \alpha\gamma\frac{d^2f}{dxdz}$$
$$+ \beta\gamma\frac{d^2f}{dydz} + \text{etc.} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{(A)}$$

By taking $\alpha$, $\beta$, $\gamma$ small enough, the sign of the right-hand side of A will depend upon that of the terms involving only the first powers of $\alpha$, $\beta$ and $\gamma$ therefore for a maximum or minimum,

$$\alpha\frac{df}{dx} + \beta\frac{df}{dy} + \gamma\frac{df}{dz} = 0$$

and since $\alpha$, $\beta$ and $\gamma$ are independent of each other, it follows that

$$\frac{df}{dx} = 0, \quad \frac{df}{dy} = 0, \quad \frac{df}{dz} = 0 \quad . \quad (B)$$

These three equations will determine the values of $x$, $y$ and $z$ which will make $f(x, y, z)$ a maximum or minimum, if such exist. By reasoning similar to that given in the last article, the conditions for a maximum or minimum are that

$$PQ > R'^2, \text{ and}$$
$$P\{PQR + 2P'Q'R' - PP'^2 - QQ'^2 - RR'^2\} > 0$$

where

$$P = \frac{d^2f}{dx^2} \qquad Q = \frac{d^2f}{dy^2} \qquad R = \frac{d^2f}{dz^2}$$
$$P' = \frac{d^2f}{dydz} \qquad Q' = \frac{d^2f}{dxdz} \qquad R' = \frac{d^2f}{dxdy}$$

For a maximum $\dfrac{d^2f}{dx^2}$ must be negative.

" minimum " " positive.

**Undetermined multipliers.**

Let $v = f(x, y, z)$ be a function of three variables, $x$, $y$, $z$, which are connected by the equation $\phi(x, y, z) = 0$ it is required to find a maximum or minimum value of $v$

For a maximum or minimum

$$\left(\frac{dv}{dx}\right)dx + \left(\frac{dv}{dy}\right)dy + \left(\frac{dv}{dz}\right)dz = 0 \quad . \quad . \quad (A)$$

and from the equation $\phi(x, y, z) = 0$ we have

$$\left(\frac{d\phi}{dx}\right)dx + \left(\frac{d\phi}{dy}\right)dy + \left(\frac{d\phi}{dz}\right)dz = 0 \quad . \quad . \quad (B)$$

Multiply equation B by $\lambda$ an arbitrary constant, and add the result to A thus

$$\left\{\left(\frac{dv}{dx}\right)+\lambda\left(\frac{d\phi}{dx}\right)\right\}dx+\left\{\left(\frac{dv}{dy}\right)+\lambda\left(\frac{d\phi}{dy}\right)\right\}dy+\left\{\left(\frac{dv}{dz}\right)+\lambda\left(\frac{d\phi}{dz}\right)\right\}dz = 0 \quad . \quad . \quad . \quad . \quad \text{(C)}$$

Since $\lambda$ is an arbitrary constant, we may give to it such a value that

$$\left(\frac{dv}{dx}\right)+\lambda\left(\frac{d\phi}{dx}\right) = 0$$

therefore C becomes

$$\left\{\left(\frac{dv}{dy}\right)+\lambda\left(\frac{d\phi}{dy}\right)\right\}dy+\left\{\left(\frac{dv}{dz}\right)+\lambda\left(\frac{d\phi}{dz}\right)\right\}dz = 0 \quad . \quad . \quad \text{(D)}$$

and since $y$ and $z$ may be considered independent variables, because $y$ and $z$ can be expressed in terms of $x$ by means of equations $v = f(x, y, z)$ and $\phi(x, y, z) = 0$, it follows that

$$\left\{\left(\frac{dv}{dy}\right)+\lambda\left(\frac{d\phi}{dy}\right)\right\}dy = 0 \quad . \quad . \quad . \quad . \quad \text{(E)}$$

$$\left\{\left(\frac{dv}{dz}\right)+\lambda\left(\frac{d\phi}{dz}\right)\right\}dz = 0 \quad . \quad . \quad . \quad . \quad \text{(F)}$$

Therefore we have the three equations

$$\left(\frac{dv}{dx}\right)+\lambda\left(\frac{d\phi}{dx}\right) = 0$$

$$\left(\frac{dv}{dy}\right)+\lambda\left(\frac{d\phi}{dy}\right) = 0$$

$$\left(\frac{dv}{dz}\right)+\lambda\left(\frac{d\phi}{dz}\right) = 0$$

together with $\phi(x, y, z) = 0$ for determining the values of $x$, $y$, $z$ and $\lambda$

**To find the rectangular parallelopiped of maximum surface that can be inscribed in a sphere whose equation is**

$$x^2 + y^2 + z^2 = r^2 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The surface is

$$S = 8(xy + xz + yz) \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $2x$, $2y$ and $2z$ are the lengths of its three coterminous edges.

By equations (1) and (2) we have

$$xdx + ydy + zdz = 0$$

and $$(y + z)dx + (x + z)dy + (y + x)dz = 0$$

for a maximum; therefore

$$y + z + \lambda x = 0 \quad . \quad . \quad . \quad . \quad . \quad (a)$$
$$x + z + \lambda y = 0 \quad . \quad . \quad . \quad . \quad . \quad (b)$$
$$x + y + \lambda z = 0 \quad . \quad . \quad . \quad . \quad . \quad (c)$$

Multiplying $(a)$, $(b)$ and $(c)$ by $x$, $y$ and $z$ respectively, and adding, we get

$$2(xy + xz + yz) + \lambda(x^2 + y^2 + z^2) = 0$$

that is, $$\frac{S}{4} + \lambda r^2 = 0$$

This determines $\lambda$ and by means of equations $(a)$, $(b)$ and $(c)$ it is evident that $x = y = z$ therefore the required rectangular parallelopiped is a cube whose edge is $2x = \frac{2r}{\sqrt{3}}$ by equation (1).

**To find the volume of the greatest rectangular block that can be inscribed in the ellipsoid whose equation is**

$$\frac{x^2}{l^2} + \frac{y^2}{m^2} + \frac{z^2}{n^2} = 1 \quad . \quad . \quad . \quad . \quad . \quad (a)$$

Let $2x$, $2y$ and $2z$ denote the lengths of three coterminous edges of the block; then its volume

$$V = 8xyz \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

From $(a)$ and $(b)$ we get by differentiating

$$\frac{2xdx}{l^2} + \frac{2ydy}{m^2} + \frac{2zdz}{n^2} = 0$$

and
$$yzdx + xzdy + xydz = 0$$

for a maximum; therefore

$$yz + \lambda\frac{x}{l^2} = 0, \quad xz + \lambda\frac{y}{m^2} = 0$$

and
$$xy + \lambda\frac{z}{n^2} = 0$$

Multiplying these three equations by $x$, $y$ and $z$ respectively, we have

$$xyz + \lambda\frac{x^2}{l^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (a)$$

$$xyz + \lambda\frac{y^2}{m^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (b)$$

$$xyz + \lambda\frac{z^2}{n^2} = 0 \quad . \quad . \quad . \quad . \quad . \quad (c)$$

and on adding $(a)$, $(b)$ and $(c)$ we have

$$3xyz + \lambda\left(\frac{x^2}{l^2} + \frac{y^2}{m^2} + \frac{z^2}{n^2}\right) = 0$$

that is,
$$\frac{3V}{8} + \lambda = 0$$

therefore
$$\lambda = -\frac{3V}{8}$$

Substituting $-\frac{3V}{8}$ for $\lambda$ in $(a)$, $(b)$ and $(c)$ we get

$$2x = \frac{2l}{\sqrt{3}}, \quad 2y = \frac{2m}{\sqrt{3}} \text{ and } 2z = \frac{2n}{\sqrt{3}}$$

therefore the maximum volume is

$$V = \frac{8lmn}{3\sqrt{3}}$$

**To find the minimum value of**

$$u = ax^2 + by^2 + cz^2$$

**subject to the conditions**

$$Ax + By + Cz = D$$

Here $du = 2axdx + 2bydy + 2czdz = 0$
for a maximum, and

$$Adz + Bdy + Cdz = 0$$

therefore

$$2ax + \lambda A = 0, \quad 2by + \lambda B = 0, \quad 2cz + \lambda C = 0$$

Hence $\quad 2(ax^2 + by^2 + cz^2) + \lambda(Ax + By + Cz) = 0$

that is, $\quad 2u + \lambda D = 0$

$$\therefore \lambda = -\frac{2u}{D}$$

Substituting for $\lambda$ in

$$2ax + \lambda A = 0 \text{ etc.}$$

we get

$$2ax = \frac{2uA}{D}$$

$$\therefore x = \frac{uA}{aD}$$

$$\therefore ax^2 = \frac{u^2A^2}{aD^2}$$

Similarly, we get $\quad by^2 = \dfrac{u^2B^2}{bD^2}$

and $\quad cz^2 = \dfrac{u^2C^2}{cD^2}$

and on adding we have

$$ax^2 + by^2 + cz^2 = \frac{u^2}{D^2}\left(\frac{A^2}{a} + \frac{B^2}{b} + \frac{C^2}{c}\right) = u$$

$$\therefore u = \frac{D^2}{\dfrac{A^2}{a} + \dfrac{B^2}{b} + \dfrac{C^2}{c}}$$

**To find the dimensions of a cistern of maximum capacity that can be formed out of 300 square feet of sheet iron, there being no lid.**

Let $x$ denote the required length, $y$ its breadth and $z$ its depth, all in feet. Then

$$xy + 2xz + 2yz = 300 = A \quad \text{say,}$$

and $u = xyz$ is to be a maximum.

From these equations we get

$$(y + 2z)dx + (x + 2z)dy + (2x + 2y)dz = 0$$

and
$$yzdx + xzdy + xydz = 0$$

for a maximum. Hence

$$y + 2z + \lambda yz = 0 \quad . \quad . \quad . \quad . \quad (a)$$
$$x + 2z + \lambda xz = 0 \quad . \quad . \quad . \quad . \quad (b)$$
$$2x + 2y + \lambda xy = 0 \quad . \quad . \quad . \quad . \quad (c)$$

Multiply $(a)$ by $x$, $(b)$ by $y$ and $(c)$ by $z$ and add; therefore

$$2xy + 4xz + 4yz + 3\lambda xyz = 0$$

that is,
$$2A + 3\lambda u = 0$$
$$\therefore \lambda = -\frac{2A}{3u}$$

Substituting for $\lambda$ in $(a)$, $(b)$ and $(c)$, we get

$$3xy + 6xz = 2A, \quad 3xy + 6yz = 2A, \quad 6xz + 6yz = 2A$$

Hence
$$x = y = 2z$$

If we substitute for $y$ and $z$ in terms of $x$ in the equation

$$xy + 2xz + 2yz = 300$$

we get
$$x = 10 = y \text{ and } z = 5$$

Therefore the cistern of maximum capacity will be 10 feet long, 10 feet broad and 5 feet deep.

**To find the triangle of minimum area which can be described about a circle of radius $r$**

The area of the triangle is $= \frac{r}{2}(a+b+c)$ and since $r$ is constant the area will be a minimum when $a+b+c$ is a minimum. It is easy to show that

$$\begin{aligned} a &= r(\cot \tfrac{1}{2}\mathrm{B} + \cot \tfrac{1}{2}\mathrm{C}) \\ b &= r(\cot \tfrac{1}{2}\mathrm{A} + \cot \tfrac{1}{2}\mathrm{C}) \\ c &= r(\cot \tfrac{1}{2}\mathrm{A} + \cot \tfrac{1}{2}\mathrm{B}) \end{aligned}$$

therefore the triangle will be a minimum when

$$2 \cot \tfrac{1}{2}\mathrm{A} + 2 \cot \tfrac{1}{2}\mathrm{B} + 2 \cot \tfrac{1}{2}\mathrm{C}$$

is a minimum.

We have also $\quad \mathrm{A} + \mathrm{B} + \mathrm{C} = 180°$

therefore by differentiating we get

$$\operatorname{cosec}^2 \tfrac{1}{2}\mathrm{A} d\mathrm{A} + \operatorname{cosec}^2 \tfrac{1}{2}\mathrm{B} d\mathrm{B} + \operatorname{cosec}^2 \tfrac{1}{2}\mathrm{C} d\mathrm{C} = 0$$

for a minimum, also

$$\begin{aligned} d\mathrm{A} + d\mathrm{B} + d\mathrm{C} &= 0 \\ \therefore \operatorname{cosec}^2 \tfrac{1}{2}\mathrm{A} + \lambda &= 0 \\ \operatorname{cosec}^2 \tfrac{1}{2}\mathrm{B} + \lambda &= 0 \\ \operatorname{cosec}^2 \tfrac{1}{2}\mathrm{C} + \lambda &= 0 \end{aligned}$$

This leads to $\mathrm{A} = \mathrm{B} = \mathrm{C}$ therefore the minimum triangle is equilateral.

*Examples.*

1. Three cubic feet of lead are to be formed into the lining of a rectangular cistern, the thickness of the lining is to be $\frac{1}{8}$ inch. Find the dimensions of the cistern sc that it may have a maximum capacity, there being no lid.

*Ans.* Length = breadth = 9·79 feet.
Depth = 4·89 feet.

2. Find a point within a triangle, such that the sum of the squares of its distances from the three angles is the least possible.

*Ans.* The c. g. of the triangle.

3. Find the minimum value of $x^3y^2z$ subject to the condition $\frac{4}{x}+\frac{5}{y}+\frac{6}{z} = 1$

*Ans.* When $\frac{3x}{4} = \frac{2y}{5} = \frac{z}{6} = 6$

therefore $x^3y^2z = 4147200$

4. Given the sum of the three edges of a rectangular block, find its dimensions such that its surface may be a maximum.

*Ans.* A cube.

5. Find the maximum value of $v$ when $v = x^2y^3z^4$ and $2x + 3y + 4z = c$

*Ans.* $\left(\frac{c}{9}\right)^9$

6. Divide a line $a$ feet long into three parts, $x$, $y$ and $z$, such that the sum of one-half the rectangle $xy$ one-third the rectangle $xz$ and one-fourth the rectangle $yz$ shall be a maximum.

*Ans.* $x = \frac{21a}{47}$, $y = \frac{20a}{47}$ and $z = \frac{6a}{47}$

## MISCELLANEOUS EXAMPLES

1. What is a differential coefficient? Give illustrations from velocity, from slope of curves, etc.

2. Prove the rule for differentiating a quotient, also for $x^m$, $\sin x$, and $\log x$.

3. Prove that $\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$

4. Differentiate $\log(\log x)$, $e^{ax}\sin rx$, $\frac{(1-x^2)^{\frac{1}{2}}\sin^{-1}x}{x}$ $x^{x^n}$, and $e^{x^x}$

5. If $x = e^{\frac{x-y}{y}}$ find $\frac{dy}{dx}$

6. Prove Taylor's Theorem. Give two examples of its use.

7. State Maclaurin's Theorem. Expand $a^x$ and $\sin x$

8. If $y = \sin kx$ find $\frac{dy}{dx}$ from first principles; also find the differential coefficient of a product.

9. Differentiate $\frac{x}{\sqrt{a^2+x^2}-x}$, $\sqrt{1-x^2}$, $\sin^{-1}x$, and $x^{\sin x}$

10. Find $\frac{d^5y}{dx^5}$ if $y = \frac{1+x}{1-x}$

11. Expand $\sin(x+a)$ in powers of $a$ by Taylor's Theorem, and afterwards put $x = 0$

12. Find the value of $\frac{e^x - e^{-x}}{\log(1+x)}$ when $x = 0$

13. If $c$ be the hypotenuse of a right-angled triangle, find the lengths of the other sides when the area of the triangle is a maximum.

14. If $y = x^n$ find $\dfrac{d^m y}{dx^m}$ $m < n$

15. Describe the meaning of $du = \left(\dfrac{du}{dx}\right)dx + \left(\dfrac{du}{dy}\right)dy$

16. If $u = \log \tan \dfrac{y}{x}$ prove that $\dfrac{d^2u}{dxdy} = \dfrac{d^2u}{dydx}$

17. Find the value of $\dfrac{x - \sin^{-1}x}{\sin^3 x}$ when $x = 0$

18. If $u$ is a function of $y$ and $z$ and these are both functions of $x$ prove that

$$\frac{du}{dx} = \left(\frac{du}{dy}\right)\frac{dy}{dx} + \left(\frac{du}{dz}\right)\frac{dz}{dx}$$

Explain the meaning of this equation. Give illustrations. If $u = z^2 + y^2 + zy$ and $z = \sin x$, $y = \epsilon^x$, find $\dfrac{du}{dx}$ in two ways.

19. Prove the rules for finding maxima and minima values of a function. Illustrate by a curve.

20. Find the strongest rectangular beam which can be cut from a cylindric tree. Find also the stiffest beam.

21. If the motion of a piston worked by a crank of length $r$ and connecting rod of length $l$ is such that its distance $x$ from the end of its stroke when the crank makes an angle $\theta$ with the line of centre, is approximately $x = r(1 - \cos\theta) + \dfrac{r^2}{4l}(1 - \cos 2\theta)$ find the acceleration when $\theta = 30°$, $r = 1$ ft., $l = 5$ ft.

22. Find the limiting value of $\dfrac{x - 1}{x^n - 1}$ when $x = 1$

23. Prove the expressions for the *subtangent*, *subnormal*, and *intercepts* on the axes by the tangent at any point to a curve.

24. Find the equations of the tangent and normal to a

curve at any point. Find these for the point (1, 6) on the curve $y^2 = 36x$

25. Expand $a^x$ by Maclaurin's Theorem. In your answer put $a = e$

26. Expand $\log(1 + x)$ and $\tan^{-1}x$

27. Find the true value of $\frac{\tan x - x}{x - \sin x}$ when $x = 0$

28. Divide 100 into two parts, so that five times the square of one part and three times the square of the other part shall be a minimum.

29. If $V = RI + \cdot 01\frac{dI}{dt}$ and if $I = 50 \sin \frac{2\pi t}{\cdot 03}$ find V and roughly indicate the nature of I and V by curves.

30. Find the area of an indicator diagram if the law of expansion is $pv =$ const. Find $p_e$ in terms of $p_1$, $p_3$, and $r$, and the amount of work per cubic foot of steam, $p_e$ being effective pressure, $p_1$ initial pressure, and $p_3$ back pressure. Find $r$ for a maximum amount of work per cubic foot of steam, assuming no condensation.

31. Find the expressions for the position of the centre and the radius of curvature of any curve at any point.

32. Find the radius of curvature at the vertex of the parabola $y = mx^2$

33. Write out the integrals of $x^m$, $\frac{1}{x-a}$, $\frac{1}{a^2+x^2}$, $e^x$, $\frac{1}{\sqrt{x^2+a^2}}$, $\frac{1}{(x-a)(x-b)}$, $f'(x) \div f(x)$

34. What is the rule for integrating by parts? Apply it in the integration of $\sqrt{a^2 - x^2}$

34*a*. Find the equation to the cycloid. Find the length of the curve.

35. Compare the volumes of a paraboloid of revolution, a cone and a cylinder on the same base and of the same height.

36. Find the shape taken by a flexible chain loaded uniformly horizontally.

37. If $V = RI + L\frac{dI}{dt}$ and $I = a \sin \frac{2\pi t}{T}$ find V

38. Find the area of the hypothetical indicator diagram, the law of the expansion being $pv^k =$ const., and afterwards the work done per cubic foot of steam. Cut off at $\frac{1}{r}$ of the stroke.

39. Find the moment of inertia of a cylinder about its axis.

40. Integrate

$$(x^2 + 7x + 12)^{-1}, \quad (1 + x + x^2)^{-1}, \quad \epsilon^{ax} \sin mx,$$
$$\sqrt{a^2 - x^2}, \text{ and } \sin mx \cos nx$$

41. Explain how we determine the most economical electrical conductor ($a$) for a constant current, ($b$) to deliver a definite amount of power at a distant place.

42. 120,000 watts are to be delivered 3 miles away; dynamo voltage 1000. Find the proper conductor. Compare the answer with answer given by Lord Kelvin's rule.

43. A straight uniform beam is fixed at one end and loaded at the other; find its shape when loaded.

44. Find $\int_0^{2\pi} \sin^2 kx dx$, $\int_0^{2\pi} \sin kx \cos rx dx$.

45. Investigate the equation to the catenary. Find the length of an arc of it, and its area.

46. Define $\sin hx$, $\cos hx$, $\tan hx$, $\sin h^{-1}x$ and $\cos h^{-1}x$.

47. Integrate

$$(1 - 2x + 2x^2)^{-1}, \quad (1 + x - x^2)^{-1}, \quad (a + b \cos \theta)^{-1},$$
$$x \log x, \quad \text{and } x^2 \epsilon x.$$

48. Find the area of the segment of an ellipse.

49. Find the surface generated by the revolution of

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}$$

about the axis of X

50. If $V = a + b \sin nt$ find I when $V = RI + L\frac{dI}{dt}$ Give the complete value of I Why do we not usually give the complete value?

51. Neglecting evanescent terms, given $V = a \sin nt$, find the current in a circuit which has resistance, self-induction and capacity.

52. Prove the rules for the development of a function in Fourier's Series.

53. If $I = a_0 + a_1 \sin pt + b_1 \cos pt + a_2 \sin 2pt + b_2 \cos 2pt +$ etc., find its mean square value.

54. Solve $\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$ Show how, as $a$ increases from 0 the nature of the solution changes. Apply to any dynamical or electrical problem.

55. State exactly what is meant by $Mdx + Ndy$ being a complete differential. What is the test? Why?

56. Integrate $xy(1 + x^2)dy - (1 + y^2)dx = 0$

$$xdy - ydx - \sqrt{x^2 + y^2}dx = 0$$

57. Integrate

$$x(1 - x^2)\frac{dy}{dx} + (2x^2 - 1)y = bx^3$$

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$$

58. Solve $D^2x + fDx + n^2x = 0$ showing the various kinds of solution for different values of $f$ and $n$ Apply the solution either to the discharging of a condenser or to the damping of vibrations of a body.

59. Prove the rule for destroying the effects due to self-induction in a branch of a network of conductors.

60. Explain what is meant by $I^2r + \frac{t^2}{r}$ in electrical problems. If I is constant, find the condition of maximum economy. What is $t$ usually, and on what assumption?

61. Operate with

$$\frac{a + b\mathrm{D} + c\mathrm{D}^2 + d\mathrm{D}^3 + e\mathrm{D}^4}{a' + b'\mathrm{D} + c'\mathrm{D}^2 + d'\mathrm{D}^3 + e'\mathrm{D}^4}$$

upon A $\sin kt$; D means $\frac{d}{dt}$.

62. Find $r$ and K so as to destroy effects of self induction, for all outside circuits. Current following any law whatever.

63. Develop the periodic function (Fig. 115).

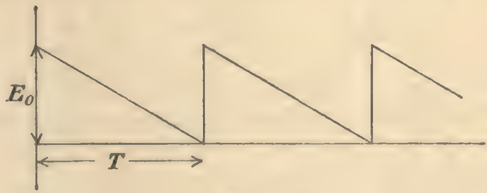


FIG. 115.

64. Integrate

$$\int\frac{dx}{1 + x - x^2}, \quad \int\frac{dx}{x^2 + 4x + 5}, \quad \int\frac{dx}{\sqrt{x^2 - ax}}$$

$$\int\frac{dx}{ax - x^2}, \quad \int e^{ax}\sqrt{\sin mx}\,dx, \quad \text{and} \quad \int e^x\sqrt{x}\,dx$$

65. Find the area of a loop of the curve $r^2 = a^2 \cos 3\theta$

66. Find the moment of inertia of a rectangle about a line through its centre parallel to one side.

67. Find the centre of area of a sector of a circle.

68. Find the law of tension in a belt on a pulley when slipping is occurring.

69. Find the law of leakage from an electric condenser through a constant resistance.

70. When is $\phi(x, y)dx + \psi(x, y)dy$ a complete differential?

71. When is $x \cos x$ a maximum or minimum?

72. Integrate (1) $\sin x \cos y dx - \cos x \sin y dy = 0$

$$(2)\ \frac{dy}{dx} + \cos xy = \sin x \cos x$$

$$(3)\ (y - x)dy + ydx = 0$$

$$(4)\ \frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

73. Investigate the motion of a body vibrating under the action of a frictional resistance.

74. What is the meaning of

$$\left(\frac{d}{dx} - a\right)^{-1}\left(\frac{d}{dx} - b\right)^{-1} X \quad ?$$

Illustrate by an example.

75. Find the tangent, normal, subtangent and subnormal to $x^m y^n = a$

76. In the curve whose equation is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, show that the part of the tangent intercepted between the axes is $a$

77. $\displaystyle\int \frac{dx}{x^2 - a^2}, \qquad \int x^2\sqrt{a + x}\,dx, \qquad \int \frac{x^3 dx}{(x-1)^2(x^2+1)}$.

78. Find the volume and surface of a paraboloid of revolution bounded by a section at right angles to the axis.

79. Find the moment of inertia of an ellipse about its major axis.

80. Prove that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its centre of mass, plus the whole mass multiplied by the square of the distance between the axes.

81. Find the moment of inertia of a circle ($a$) about its centre, ($b$) about a diameter.

82. Find the moment of inertia of a sphere ($a$) about its centre, ($b$) about a diameter.

83. Find the moment of inertia of a right cylinder about its axis.

84. Solve $(2x + 4y + 7)dx + (x + 7y + 1)dy = 0$.

85. Neglecting evanescent terms, given $V = a \sin nt$, find the current in a circuit which has resistance R, capacity K, and self-induction L.

86. Develop the periodic function.

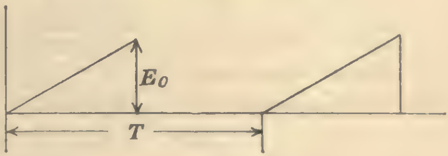


FIG. 116.

87. Express the motion of the piston of a steam engine approximately as two simple harmonic motions.

88. With the hypothetic law, work per cubic foot of steam $= 144r\left\{p\left(\frac{1 + \log r}{r}\right) - p_3\right\} - x$ If Mr. Willans' best law of cut-off, $r = \frac{p'}{p_3 + 10}$ is correct, find how $x$ depends on $r$

89. Show that when a curve is very *flat*, $\frac{d^2y}{dx^2}$ is its *curvature.*

90. Solve $\frac{d^2y}{dx^2} + f\frac{dy}{dx} + 25y = 0$

1st. When $f$ is greater than 10 say $f = 12$
2nd. When $f$ is equal to 10
3rd. When $f$ is less than 10 say $f = 8$, or 6, or 4, or 2, or 0.

91. Plot the three kinds of curves you obtain in Example 90.

92. Find the area of a cycloid, the radius of the generating circle being $r$.

93. Find expressions for the position of the centre of curvature of a curve.

94. For what curves is it true that the subnormal is constant?

95. Expand $\log(x + h)$ and put $x = 1$

96. Integrate $(x^2 - 36)^{-1}$, $\sin^2 x$, and $\cos^5 x$

97. If $y = x(20 - x)$ for what value of $x$ will $y$ have its greatest value? If $\frac{dy}{dx} = ky$, show that $y = ae^{kx}$

98. Show by actually plotting points on squared paper that if we add the ordinates of $y = a \sin x$ and $y = b \cos x$ we obtain the curve $y = \sqrt{a^2 + b^2} \sin(x + e)$ where $e = \tan^{-1}\frac{b}{a}$

99. Write in words the meaning of

$$\frac{d^2s}{dt^2} = -as; \quad \frac{d^2s}{dt^2} + f\frac{ds}{dt} + as = 0$$

100. Find the maximum and minimum values of

$$\frac{(a+x)(b+x)}{x}, \quad \frac{9x^2+6x-7}{12x-8}, \quad \text{and} \quad \frac{a^2}{x} + \frac{b^2}{a-x}$$

101. Find two factors of $a$ so that the sum of their squares is a minimum.

102. Solve $(3x - 7y + 7)dy + (7x - 3y + 3)dx = 0$

103. If $x$ is the distance of the piston of a steam engine from mid-stroke, $\theta$ angle of crank from dead point, $r$ length of crank, neglecting angularity of connecting rod, $x = r\cos\theta$ Take $\theta = \frac{2\pi t}{T}$ where $T$ = time in seconds of one revolution. If weight of piston is 200 lb., $r = 1$ foot, make a diagram showing forces acting on piston rod due to acceleration of piston.

104. If $l$ is length of connecting rod, the motion of piston may be taken as very nearly being one simple harmonic motion plus a much smaller one of half the period. Or $x = r\cos\theta - \frac{r^2}{4l}(1 - \cos 2\theta)$ Now find the diagram of accelerating forces if $l = 4$ feet. Observe that the overtone or "kick" becomes more important in the acceleration than it was in the mere motion.

105. Find the subnormal and subtangent of the curve whose equation is $y^n = a^{n-1}x$

106. Show that if $s = ae^{-pt}\sin nt$ then

$$\frac{d^2s}{dt^2} + 2p\frac{ds}{dt} + (p^2 + n^2)s = 0$$

107. When the rate of increase of a quantity is proportional to the quantity, or

$$\frac{dy}{dx} = ky$$

show that

$$y = ae^{kx}$$

This has been called the compound interest law. Show that it is the law for the sections of long pump-rods, etc.; law for leaking electric condensers; law for belts slipping on pulleys; law for atmospheric pressure as one changes in level; and many other cases.

108. If

$$\frac{d^2y}{dx^2} + n^2y = 0$$

$$y = \mathrm{A}\sin nx + \mathrm{B}\cos nx$$

where A and B are any constants whatever.

109. Find the area of the curve $y = ax^n$ between the ordinates $x = x'$ and $x = x_2$ Show the shape of this curve when $n = 2$ when $n = \frac{1}{2}$ and when $n = -1{\cdot}1$ The answer is indeterminate when $n = -1$ What is then the area?

110. Find the volume and surface of a sphere.

111. Integrate $\tan x$, $\frac{x^3}{a+bx^4}$, $\frac{x}{\sqrt{1-x^2}}$, $\frac{1}{x+bx^2}$, $\frac{(1+x^2)^2}{x^2}$, $\frac{x-2}{x\sqrt{x}}$, $\tan^2 x$, $\sqrt{\frac{a+x}{a-x}}$, $x\sqrt{x+a}$, $\frac{1}{\sqrt{x+a}+\sqrt{x}}$, $\frac{x^3}{\sqrt{a^8-x^8}}$, $\frac{1}{x\log x}$, and $\frac{x^2}{(a+bx)^3}$.

112. Integrate $\frac{1}{x^2+px+q}$ show that we obtain quite different answers for certain values of $p$ and $q$

113. Integrate $\frac{ax+b}{x^2+px+q}$, $(5x^2+4x+8)^{-1}$, and compare $\int\frac{dx}{1-a^2x^2}$ and $\int\frac{dx}{1+a^2x^2}$

114. Integrate by parts—

$$x\log x\,,\quad \sqrt{a^2-x^2}\,,\quad \text{and}\quad a^5e^{ax}$$

115. If $s = a + bt + ct^2$ find velocity and acceleration. What are $a$, $b$, and $c$? If $s = a\sin pt$ find velocity and acceleration. What do $a$ and $p$ mean? Deduce the rule for finding the periodic time in simply vibrating bodies.

116. Find the result of the operation—

$$\frac{\mathrm{A}+\mathrm{BD}+\mathrm{CD}^2+\mathrm{ED}^3}{a+b\mathrm{D}+c\mathrm{D}+d\mathrm{D}^3}$$

upon $m\sin pt$ Use this in finding the current in a circuit whose E.M.F. is $m\sin pt$, with resistance R self-induction L and having in series a condenser of capacity K If there is no E.M.F. find the current (that is, find the surging current when the condenser is discharged through this circuit).

117. Prove Taylor's Theorem. Expand $\sin x$ by Maclaurin's Theorem.

118. Illustrate $\frac{d^2u}{dxdy} = \frac{d^2u}{dydx}$ by taking $u = \log\left(\tan\frac{y}{x}\right)$

119. What are the rules for finding maxima and minima values of a function of one variable? Illustrate by curves. The volume of a cylinder being fixed, find its height $h$ and radius $r$ when the sum of the areas of its convex surface and one end is a minimum.

120. Find the tangent through the point (1 , 5) to the curve $y = ax^4$

121. Find the curvature at the origin in the curve $y = ax^4$

122. A straight beam of uniform section, fixed at one end and loaded uniformly. Find its shape.

123. Integrate

$$2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 15y = 0$$

and

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$$

124. A man sitting in a train says: "We are now going at sixty miles an hour." What does he mean? Illustrate your answer by a curve.

125. Find the square root of the mean square of $a \sin pt$

126. Compare the areas of a parabolic segment, a rectangle, and a triangle on the same base and with same height.

127. Prove that a chain loaded uniformly horizontally is a parabolic curve.

128. What curves have their subtangents $= x^2$?

129. Find by integration the centre of gravity of a segment of a paraboloid of revolution whose base is perpendicular to its axis.

130. Find the moment of inertia of a circular cylinder about a diameter of one of its ends.

131. Integrate $\dfrac{3x - 1}{x^3 + x^2 - 2x}$ and $\dfrac{1}{a + bx + cx^2}$

132. Integrate

$$\frac{1 + x}{1 + y} + \frac{dx}{dy} = 0, \qquad y + \sqrt{x^2 + y^2} = x\frac{dy}{dx}$$

133. State the rule for solving linear differential equations with constant coefficients. Find $y$ if

$$(D-\alpha)(D-\beta)(D-\gamma)y = 0$$

134. Find the volume and surface generated by the revolution of an equilateral triangle about a line passing through its vertex parallel to its base.

135. If the law of expansion of steam in the cylinder of a steam engine be $pv^k =$ const., the work done per stroke in foot-pounds is

$$al\left\{p'\frac{kr^{-1}-r^{-k}}{k-1}-p_3\right\}$$

where $l$ is the stroke in feet, $a$ the cross-section of the piston in square inches, $p'$ the initial pressure, $p_3$ the back pressure, the cut-off taking place at one $r$th of the stroke. Prove this. What is the true value when $k = 1$?

136. Find the volume generated by a cycloid revolving about its base.

137. Find the moment of inertia of an ellipse about a diameter.

138. Find the shape assumed by a uniformly heavy telegraph wire hanging between two points of support.

139. Solve $\dfrac{d^2y}{dx^2}+l^2y = \cos mx$

140. Solve $\dfrac{d^4y}{dx^4}-2a\dfrac{dy}{dx}+a^2y = e^x$

# INDEX

THE END

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